**Ratios in Triangles and Trapezoids**

Consider this figure of a triangle ABC and a segment DE from line AB to line AC.

This figure can be the basis of a number of problems and theorems. First, consider a couple of familiar scenarios:

**Start with equal ratios of triangle sides**
- Given that \( \frac{AD}{AB} = \frac{AE}{AC} = k \), then triangle ABC is similar to triangle ADE, so \( \frac{DE}{BC} = k \) also and the corresponding angles are equal. In addition, line DE is parallel to line BC.

**Start with parallels: either of these**
- Given that DE is parallel to BC, if we denote \( \frac{AD}{AB} \) by \( k \), then also \( \frac{AE}{AC} = k \) and \( \frac{DE}{BC} = k \).
- Given that DE is parallel to BC, if we denote \( \frac{DE}{BC} \) by \( k \), then also \( \frac{AE}{AC} = k \) and \( \frac{AD}{AB} = k \).

**Numerical examples**

In any of these cases, let \( k = \frac{4}{5} \). Then if we know \( AB = x \), then \( AD = \frac{4}{5}x \). If we know \( AD = u \), then \( AB = \frac{5}{4}u \). Similar reasoning holds for AC and AE.

**Altitudes**

The ratio of altitudes of ABC and ADE is the same as for the sides. Let H be on line BC so that AH is perpendicular to BC, so AH is the altitude of ABC through A. Also let H intersect line DE at G, so AG is the altitude of ADE through A. Then show that triangle ABH is similar to triangle ADG, so \( \frac{AG}{AH} = \frac{AD}{AB} = \frac{k}{G} \).

**A new element: What about DB or EC?**
- In either case, if \( \frac{AD}{AB} = k \), then since \( AD + DB = AB \), then \( \frac{AD + DB}{AB} = 1 \), so \( k + \frac{DB}{AB} = 1 \) and \( DB/AB = 1 - k \). Likewise \( EC/AC = 1 - k \).

**Numerical examples**
- In any of these cases, if \( k = \frac{4}{5} \), then \( AD = \frac{4}{5}AB \), so the rest of the segment = \( (1/5)AB \). This illustrates \( DB/AB = 1 - k = 1 - (4/5) = 1/5 \).
- This reasoning even when \( AD > AB \), so \( k > 1 \). (In this case B is on segment AD.) For example, suppose \( AD/AB = \frac{4}{3} \). Then \( DB/AB = 1 - (4/3) = -1/3 \). The sign is correct, for then AB and DB have opposite direction.
**New variant: Start with the figure BCDE and work up!**

Suppose you start with the bottom of the previous figure: the trapezoid BCED. Given this figure, it is easy to construct A with the straightedge. Just draw lines BD and CE and intersect them. If A is determined, it must be possible to compute the distance BA from B to A if we know some information about BCED.

How can this be done? Just reverse some of the arguments before. We begin this time with the numerical examples.

- Assume that DE is parallel to BC and also \( DE/BC = 4/5 \). This is the same \( k \) as before. Suppose we know \( DB = p \), what is \( AB \)? In this case we have already seen that \( DB/AD = 1 – k = 1/5 \). So \( AD = 5 \ DB = 5p \).
- What is \( AD \)? \( AD = AB - DB = 5p - p = 4p \). This is pretty easy to see intuitively. Since \( AD = (4/5) \ AB \) and \( DB = (1/5) \ AB \), then \( AD/DB = (4/5)/(1/5) = 4! \)

So the conclusion is that **we can figure out exactly where A is located on line BD**. A is the unique point on line BD with signed ratio \( BA/BD = 5 \).

**Altitudes**

Suppose we know the height of trapezoid BCED and the ratios of the parallel sides. Can we find the height to triangle ABC? Yes!

- By the same reasoning with ratios we have \( AH = 5GH \), so the height of triangle ABC is 5 times the height of the trapezoid BCED!

**IMPORTANT**. This says that if the height of the trapezoid \( GH = x \), then the height of ABC (and the distance from A to line BC) is 5\( x \) regardless of the position ("left or right") of DE or the angles at B and C. The only determining factors are the height GH and the ratio \( DE/BC \).

If we make this figure with Sketchpad and drag DE so that the length is constant and the height \( x \) is constant, then A will move at a distance of 5\( x \) from BC and thus along a line parallel to BC.

**General, non-numerical solution**

For clarity, we assumed above that \( k = 4/5 \). Instead, if we just assume that \( DE/BC \) is some ratio \( k \), then we saw before that \( DB/AB = (1-k) \). So you can deduce \( AB/DB \) from this. The same ratio will apply to \( AH/GH \).

**Test Your Understanding**: Draw a figure BCDE on graph paper with some simple K. Predict the height of A and then check by the coordinates on the graph paper.