3.1 Tangent circles

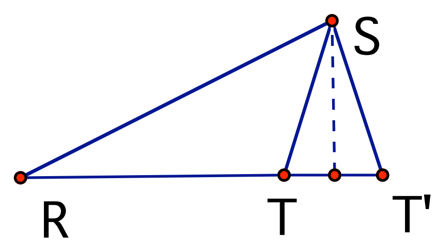
--- Part (a): Given an angle ABC, a circle is tangent to rays BA and BC if and only if the center P of the circle is on the angle bisector of ABC.

Proof: (if) If the circle is tangent, then P is on the angle bisector.

Suppose the circle is tangent to BA at D and tangent to BC at E. If the circle is tangent, then PD is perpendicular to BA, so triangle PDB is a right triangle with right angle at D. Likewise, triangle PEB is a right triangle with right angle at E, so angle BDP = angle BEP.

Since, PD and PE are radii of the same circle, they are congruent. Also the segment BP is common to both, it is the hypotenuse of each right triangle.

We can conclude that triangle BDP is congruent to triangle BEP if we can use the criterion SSA. In general this is not a valid criterion, since there are two triangles RST and RST' with the same SSA, with supplementary angles at T and T' (as shown). But if the angle at T is a right angle, then T = T' and there is only one triangle with this SSA data. (Problem 2.2)



Thus triangle PDB is congruent to triangle PEB by SSA for right triangles; and thus angle PBD = angle PBE. This means that BP bisects angle ABC.

Proof (only if): P on the angle bisector implies P is the center of a circle tangent to both rays.

Let points D and E be the feet of the perpendiculars from P to the rays BA and BC. We are given that angle PBD = angle PBE. We also know angle BDP = angle BDP = right angle. Finally, the segment BP = BP. Thus triangle BPD = triangle BPE by AAS.

Therefore PD = PE. If we construct the circle with center P through D, then is also passes through E. The circle is tangent to both rays because these radii are perpendicular to the respective rays.

--- Part (b): The bisectors of the interior angles of any triangle ABC are concurrent.

Proof: For any point D inside the triangle, one can construct circles c1, c2, c3 with center D that are tangent to BC, CA, AB, respectively. Let the radii of these circles be D1, D2, D3. Then by (a), D on the angle bisector of A if and only if D2 = D3, and D on the angle bisector of B if and only if D3 = D1.

Now let D be chosen to be the intersection of the angle bisectors of A and B. Then we have seen that all 3 radii are equal, since D2 = D3 and D3 = D1. But then D1 = D2, so I is also on the angle bisector of angle C. This point is thus a point of concurrence of all three angle bisectors. (It is usually denoted by I and is called the incenter of ABC.)

--- Part (c): For any triangle there is exactly one circle inscribed in the triangle.

Proof: The center of any such circle, being tangent to each pair of sides, must lie on the bisector of each vertex angle of triangle ABC, by (a). Thus the center of the circle must be I, from (b); and the radius = D1 = D2 = D3 as in (b). So there is only one circle.

3.2 Distance locus for intersecting lines

------ Part (a): If the distance from a point P to a line m is defined as the distance from point P to the closest point Q on the line, tell how to find this closest point and prove that this is the correct point.

Claim: Let Q be the point on m so that PQ is perpendicular to m. For any other point A on m, distinct from Q, PA > PQ. Thus the distance from P to m equals the length of PQ, and Q is the closest point. Q can be constructed as the intersection of m with the line through P perpendicular to m.

Proof: For any point A on m distinct from Q, the triangle PQA is a right triangle with right angle at Q. We will prove that PQ < PA.

First, we observe that in triangle PQA, the angles at P and A are acute. We can see this from the triangle angle sum or from the exterior angle inequality. For the latter, observe that the exterior angle of PQA at Q is a right angle, so this right angle is greater than the interior angles at P and A.

Thus in triangle PQA, angle Q is greater than angle A. By the "bigger side – bigger angle" inequality, we conclude PA > PQ.

----- Part (b): Given two intersecting lines, tell what is the set of points equidistant from each line and prove your assertion.

Claim: The set of points equidistant from two intersecting lines g and h is a pair of perpendicular lines, each of which bisect a pair of vertical angles formed by g and h.

Set-up: Let g and h be two lines intersecting in a point O. Let P be any point.

By 3.2a above, there is a point G on g and a point H on h so that PG is perpendicular to g and PH is perpendicular to h. PG is the distance to g and PH is the distance to h.

Claim 1. If P is distinct from O and equidistant from g and h, then P is on the angle bisector of GOH.

Proof: We are given PG =PH, since P is equidistant from the lines. Let c be the circle with center P and radius PG = PH. The circle passes through G and H. Since the radii PG and PH are perpendicular to the two rays, respectively, the circle is tangent to g and h. So by Problem 3.1a above, P is on the angle bisector of GOH.

Special case: Note that neither G nor H can be O if PG = PH. If for example G = O, then the triangle PGH is a right triangle and PG > PH.

Claim 2. If P is on the angle bisector of one of the angles at O formed by g and h, then P is equidistant from g and h.

Proof: By problem 3.1a above, such a P is the center of a circle tangent to g and h. The feet G and H of the perpendiculars from P to the lines are then radii of the circle and thus equal. But PG = PH = distance from P to the lines, by 3.2a.

Proof of Main Claim: Putting these two together, if P is equidistant from g and h, then either P = O (when the distance = 0) or P is on one of the four bisectors of angles formed by g and h. These four bisectors are rays that, with O, form two perpendicular lines. Conversely, if P is on this pair of lines, then by Claim 2 P is equidistant from g and h.