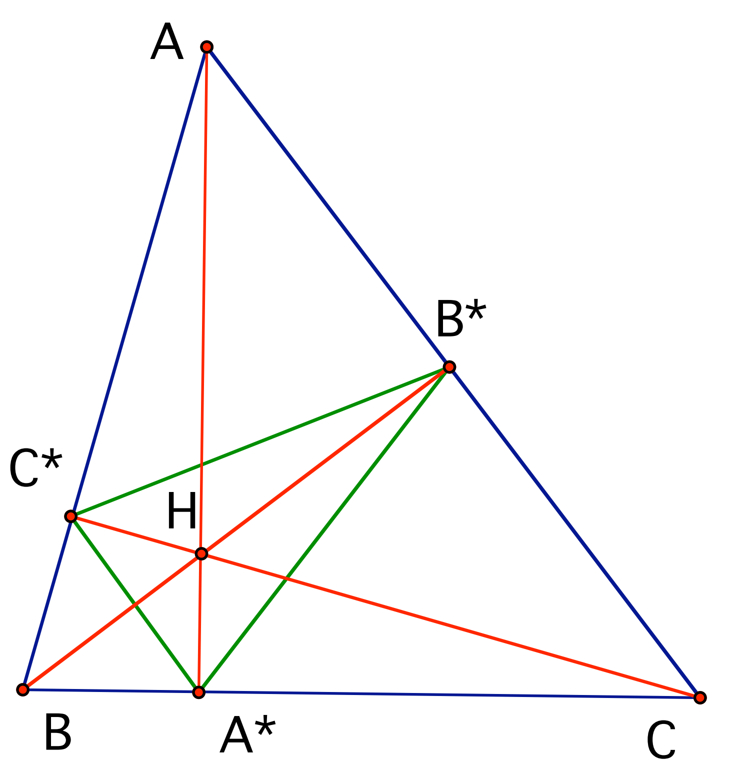
Altitudes and the Orthic Triangle of Triangle ABC

Given a triangle ABC with acute angles, let A*, B*, C* be the feet of the altitudes of the triangle: A*, B*, C* are points on the sides of the triangle so that AA* BB*, CC* are altitudes.

Then we have proved earlier that the altitudes are concurrent at a point H. (The proof used the relationship between the perpendicular bisectors of the sides of a triangle and the altitudes of its midpoint triangle).

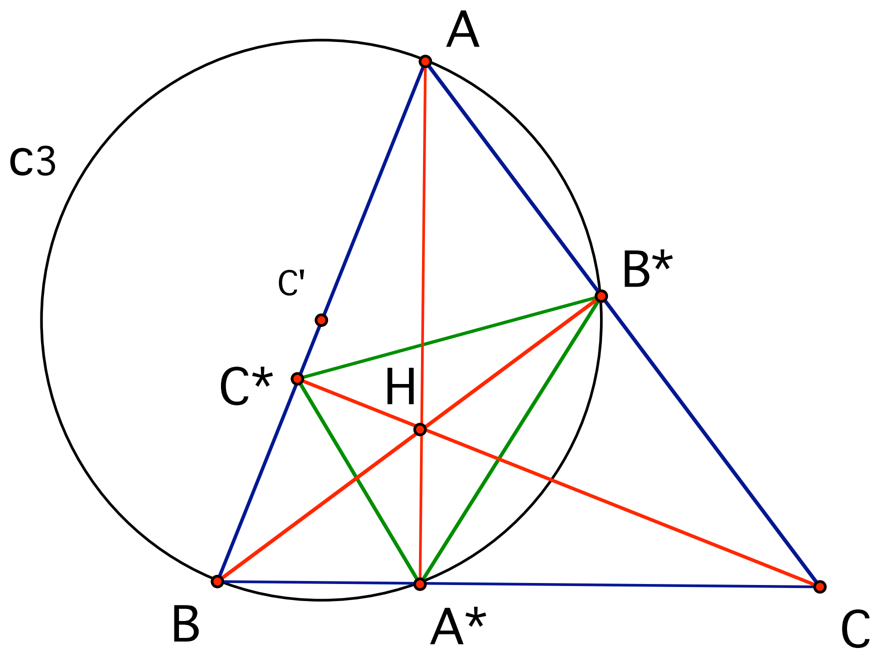
The **orthic triangle** of ABC is defined to be A*B*C*. This triangle has some remarkable properties that we shall prove:

1. The **altitudes and sides of ABC are interior and exterior angle bisectors of orthic triangle A*B*C***, so H is the incenter of A*B*C* and A, B, C are the 3 ecenters (centers of escribed circles).
2. The **sides of the orthic triangle form an "optical" or "billiard" path** reflecting off the sides of ABC.
3. From this it can be proved that the orthic triangle A*B*C* has the **smallest perimeter** of any triangle with vertices on the sides of ABC.



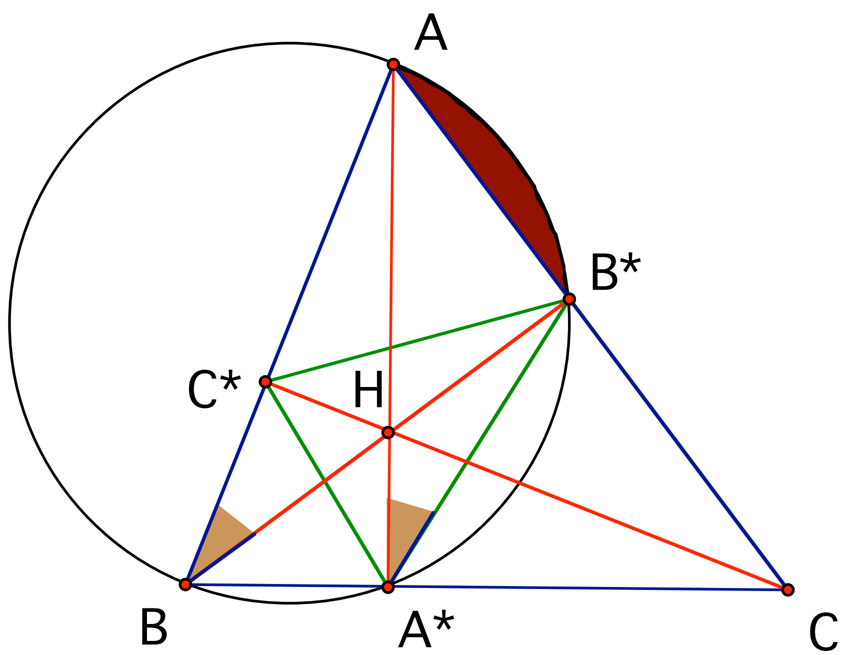
Part 1: Prove that the altitudes and sides of ABC are angle bisectors of A*B*C*

Lemma 1. Continuing with the same figure, the circle c_3 with diameter AB intersects AC at B* and BC as A*.



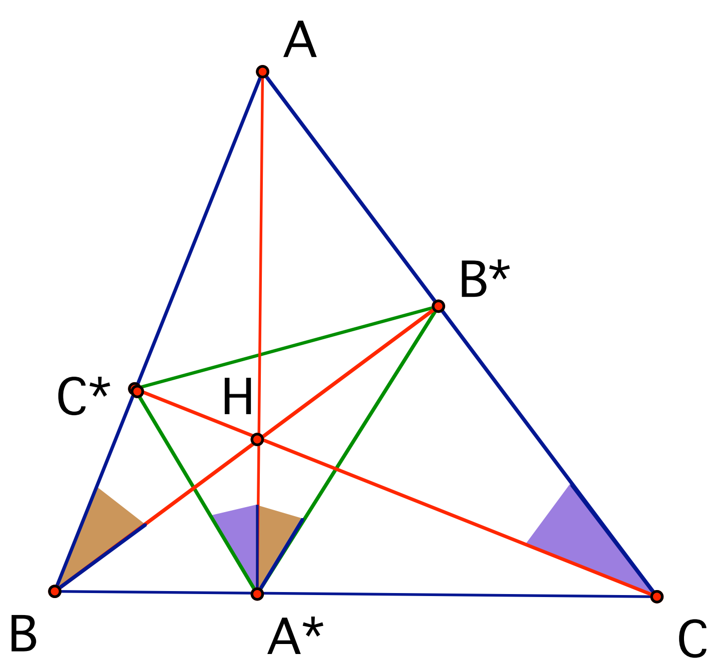
Proof. The center of the circle is the midpoint C' of AB. By the inscribed angle theorem (Carpenter theorem), since AC'B is a diameter and a straight angle, for any point P on c_3, the angle APB is a right angle. Thus the circle intersects AC at a point P so that BP is perpendicular to AC; the only such point is P = B*. Likewise, the circle intersects BC at A*.

Lemma 2. Continuing with the same figure, angle ABB* = angle AA*B*.



Proof: Both angles are angles inscribed in circle c with diameter AB. They both equal half the arc angle of arc B*A. Thus they are equal.

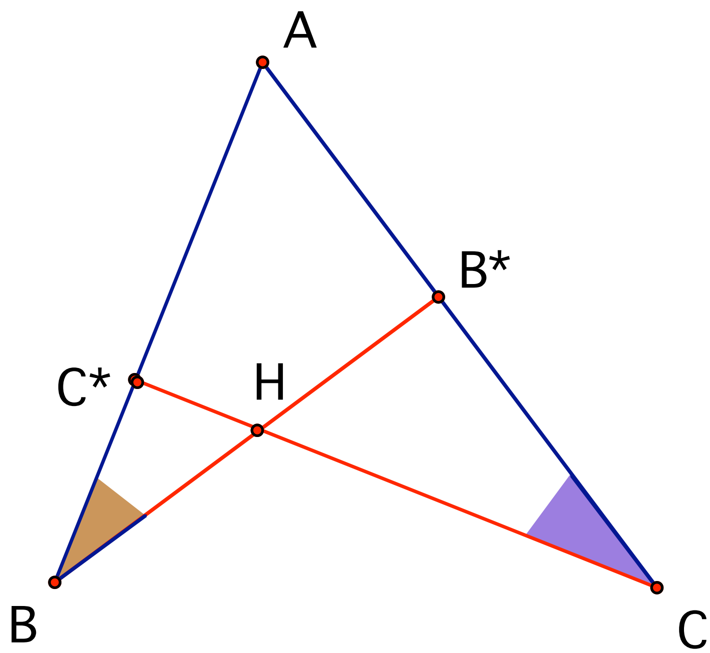
Corollary. Continuing with the same figure, angle ACC* = angle AA*C*.



Proof: Just replace B with C in the Lemma 2.

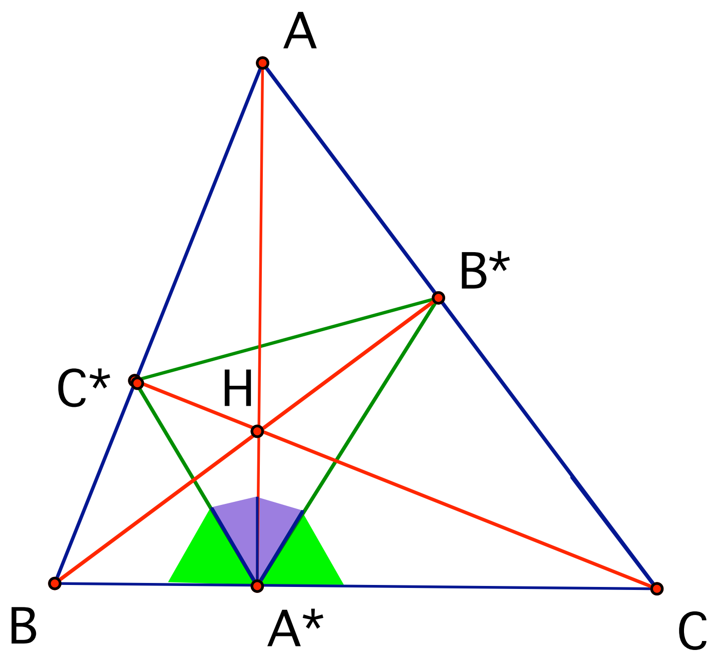
Lemma 3. Continuing with the same figure, angle AA*C*= angle AA*B*. In other words A*A bisects angle A* of triangle A*B*C*.

Proof. We have seen already from Lemma 2 that angle AA*B*. = angle ABB* and angle AA*C*. = angle ACC*.



But angle ABB* = angle ACC* by similar triangles. Both triangles ABB* and ACC* are right triangles with right angles at B* and C* and a shared angle at A, so by AA, triangles ABB* is similar to triangle ACC* and thus the angles are equal.

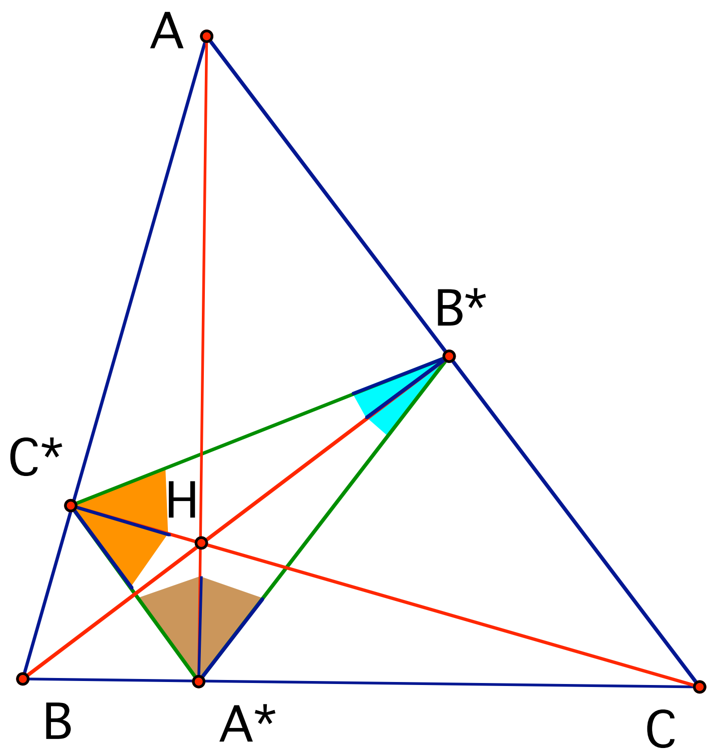
Corollary: In the figure above, angle C*A*B = angle B*A*C and line BC bisects the exterior angles at A* of triangle A*B*C*.



Proof: The exterior angle bisector at A* is the line through A* perpendicular to the interior angle bisector, which was proved to be A*A. Thus BC is this line.

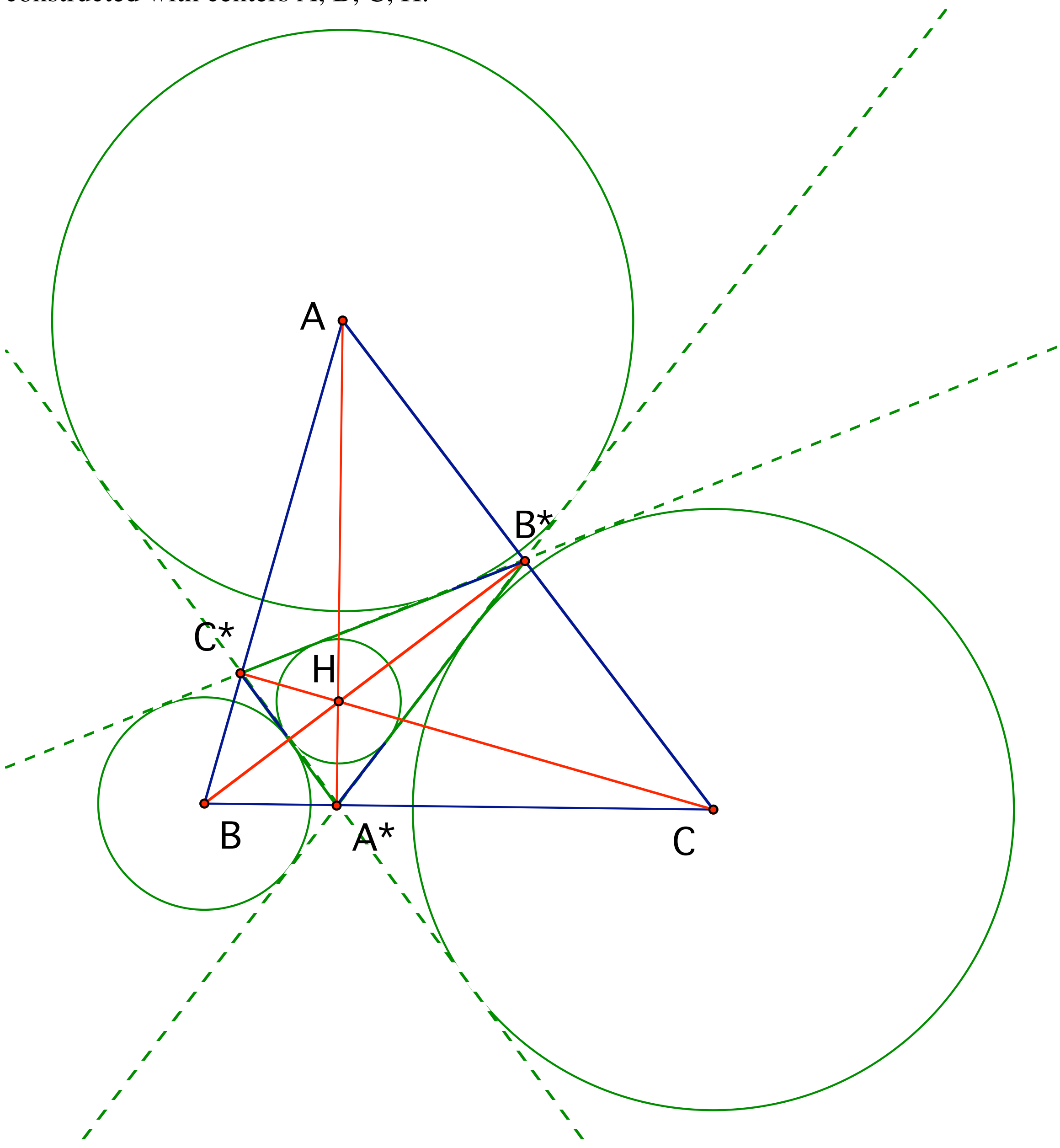
If we set x = angle AA*C*= angle AA*B*, then angle C*A*B = 90 – x = angle B*A*C. Each angle is also half of an exterior angle obtained by extending a side of A*B*C*.

Theorem: If A*B*C* is the orthic triangle of ABC, then the altitudes of ABC bisect the interior angles of A*B*C* and the sides of ABC bisect the exterior angles.



Proof. This was proved for vertex A* in Lemma 3 and its Corollary. Since A* could be chosen to be any vertex of A*B*C*, this proves the theorem for the vertices at B* and C* by the same reasoning.

Corollary: The orthocenter H of ABC is the incenter of A*B*C*, and A, B and C are the ecenters of A*B*C*. Thus four circles tangent to lines A*B*, B*C*, C*A* can be constructed with centers A, B, C, H.



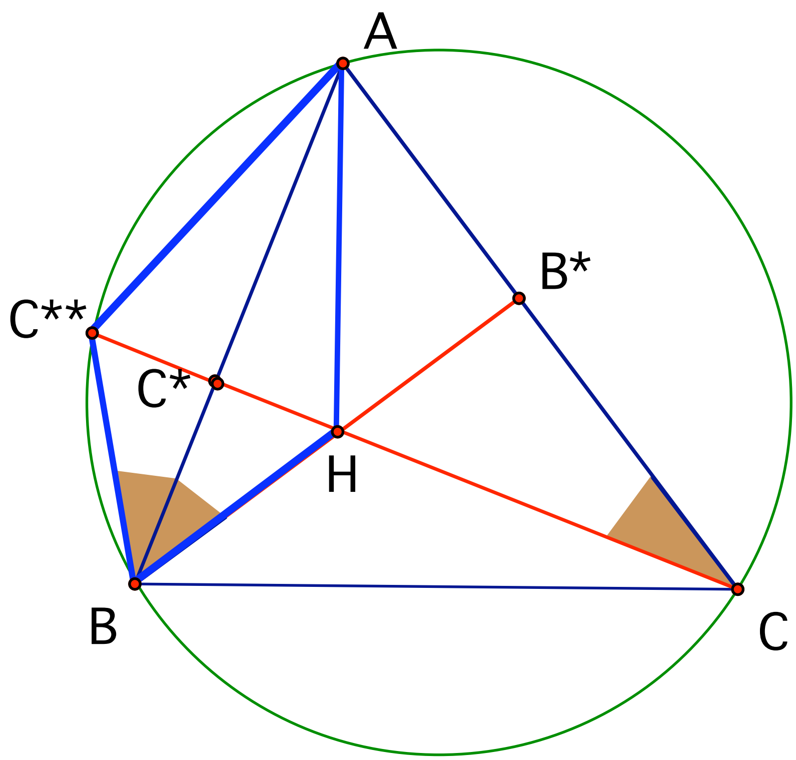
Relation between the Orthocenter and the Circumcircle

The triangle ABC can be inscribed in a circle called the circumcircle of ABC. There are some remarkable relationships between the orthocenter H and the circumcircle.

The altitude line CC* intersects the circumcircle in two points. One is C. Denote the other one by C**.

Proposition. The point CC* is the reflection of H in line AB.

This implies that the figure HBC**A is a kite, and C* is the midpoint of H and C**.



Proof: We have seen in Lemma 3 above that the triangles ABB* and ACC* are similar, so that angle ABB* is congruent to angle ACC*.

But angle ACC* is the same angle as angle ACC** is the same angle as angle C*BC**. Angle ABB* is the same angle as angle ABH is the same as angle C*BH.

Angle ACC** is an inscribed angle subtending the same arc as angle ABC**, so these two angles are equal. Thus all 3 angles are congruent: angle C*BH = angle ACC* = angle C*BC**.

Applying this proposition to each altitude, we get this theorem.

Theorem. Given an acute triangle ABC inscribed in a circle c. Let A**, B***, C*** be the intersections of the altitudes of ABC with the circle (besides A, B, C, which are also intersections). Then these points are reflections of H in the sides of ABC and triangle A**B**C** is similar to the orthic triangle A*B*C*. In fact the dilation with center H and ratio 1/2 takes A**B**C** to A*B*C*.