Quiz 8 ANSWER SHEET

Question 1

Let the vector field $F = (1, z + x, y)$.

Let the triangle T in (x, y, z)-space consist of the points in the first octant satisfying equation $\frac{x}{2} + \frac{y}{3} + z = 1$ (i.e., the points satisfy the equation plus the inequalities $x \geq 0$, $y \geq 0$, $z \geq 0$).

(a) Set up as a double integral ready to evaluate (but do *not* evaluate) this surface integral, where T is oriented by the normal pointing in the direction opposite from the origin, i.e, upward orientation.

$$\int\int_T \operatorname{curl} F \cdot n \, dS$$

Note: The answer should be a "plain vanilla" double integral, with no vector stuff at all left in it.

Answer: First, compute $\operatorname{curl} F = \mathbf{k}$.

A normal to the plane is the gradient of $\frac{x}{2} + \frac{y}{3} + z$, which is $(\frac{1}{2}, \frac{1}{3}, 1)$. This is pointing in the direction of increasing values of $\frac{x}{2} + \frac{y}{3} + z$, which is opposite the origin. (The other choice would be $(-\frac{1}{2}, -\frac{1}{3}, -1)$, which is pointing to the half-space containing the origin.

Compute the vertices of the triangle by solving equations such as $\frac{x}{2} + \frac{y}{3} + z = 1$ and $y = 0$ and $z = 0$, etc., to get vertices $(2, 0, 0)$, $(0, 3, 0)$, $(0, 0, 1)$.

Answer by using (x, y) as parameters: The plane can be viewed as the graph of the function $z = r(x, y) = 1 - \frac{x}{2} - \frac{y}{3}$.

The region in (x, y)-plane that maps to the triangle is the triangle with vertices $(2, 0)$, $(0, 3)$, $(0, 0)$ obtained by projecting to the $(x, y, 0)$-plane in the z direction. So this is the triangle that will be the domain of integration in the parameter space.

The integrand will be $\operatorname{curl} F \cdot K$, where $K = \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = (\frac{1}{2}, \frac{1}{3}, 1)$ or $K = -(\frac{1}{2}, \frac{1}{3}, 1)$, whichever agrees with the given orientation of the plane. Clearly in this case, $K = (\frac{1}{2}, \frac{1}{3}, 1)$.

So $\operatorname{curl} F \cdot K = (0, 0, 1) \cdot (\frac{1}{2}, \frac{1}{3}, 1) = 1$, and the integral over the triangle with vertices $(2, 0)$, $(0, 3)$, $(0, 0)$ is

$$\int_0^2 \int_0^{3 - \frac{3}{2}x} 1 \, dy \, dx = 3$$

Answer by using (x, z) as parameters: The plane can be viewed as the graph of the function $y = r(x, z) = 3 - \frac{3}{2}x - 3z$.

The region in (x, z)-plane that maps to the triangle in the problem is the triangle with vertices $(2, 0)$, $(0, 0)$, $(0, 1)$ obtained by projecting to the $(x, 0, z)$-plane in the y direction. So this is the triangle that will be the domain of integration in the parameter space.

The integrand will be $\operatorname{curl} F \cdot K$, where $K = \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial z} = (-\frac{3}{2}, -1, -3)$ or $K = -(-\frac{3}{2}, -1, -3) = (\frac{3}{2}, 1, 3)$, whichever agrees with the given orientation of the plane. Clearly in this case, $K = (\frac{3}{2}, 1, 3)$ is the choice, since it is a positive multiple of the orientation normal given in the problem: $(\frac{1}{2}, \frac{1}{3}, 1)$..

So $\operatorname{curl} F \cdot K = (0, 0, 1) \cdot (\frac{3}{2}, 1, 3) = 3$, and the integral over the triangle with vertices $(2, 0)$, $(0, 0)$, $(0, 1)$ is

$$\int_0^2 \int_0^{1 - \frac{1}{2}x} 3 \, dx \, dz = 3$$

Question 2: For the same F and T, use Stokes' Theorem to write a line integral that will be equal to the surface integral above. Again, in this case, write the integral as an integral (or sum of integrals) ready to evaluate (but do *not* evaluate).

Note: The answer should be a "plain vanilla" single integral, with no vector stuff at all left in it.

Answer: The line integral $\int_C F \cdot d\mathbf{r}$ will be over a boundary C consisting of 3 parts, the integral on segment C_1 from $(2,0,0)$ to $(0,3,0)$, on segment C_2 from $(0,3,0)$ to $(0,0,1)$, and on segment C_3 from $(0,0,1)$ to $(2,0,0)$. The process for each one is more or less the same, so we will give extra detail for the first and less for the others.

Line Integral on C_1 by a couple of methods:

Parametrizing by x: The segment from $(2,0,0)$ to $(0,3,0)$ is just a segment in the (x,y)-plane that can be parametrized by x running from 2 down to 0, thus:

$$(x, -\frac{3}{2}x + 3, 0), \quad 2 \geq x \geq 0$$

Since $F(x, -\frac{3}{2}x + 3, 0) = (1, x, -\frac{3}{2}x + 3)$, the line integral then becomes

$$\int_2^0 (1\,dx + x\,d(-\frac{3}{2}x + 3) + 0) = -\int_0^2 (1 - \frac{3}{2}x)\,dx = -(x - \frac{3x^2}{4})|_0^2 = 1$$

Parametrizing by y: The segment from $(2,0,0)$ to $(0,3,0)$ can be parametrized by y running from 0 to 3, thus:

$$(-\frac{2}{3}y + 2, y, 0), \quad 0 \leq y \leq 3$$

Since $F(-\frac{2}{3}y + 2, y, 0) = (1, -\frac{2}{3}y + 2, y)$, the line integral then becomes

$$\int_0^3 1\,d(-\frac{2}{3}y + 2) + (-\frac{2}{3}y + 2)\,dy + 0 = \int_0^3 (\frac{4}{3} - \frac{2}{3}y)\,dy = (\frac{4}{3}y - \frac{1}{3}y^2)|_0^3 = 1$$

Parametrizing by t: The segment from $(2,0,0)$ to $(0,3,0)$ can be parametrized by t running from 0 to 1, thus:

$$(1-t)(2,0,0) = t(0,3,0) = (2 - 2t, 3t, 0), \quad 0 \leq t \leq 1$$

Since $F(2 - 2t, 3t, 0) = (1, 2 - 2t, 3t)$, the line integral then becomes

$$\int_0^1 1\,d(2 - 2t) + (2 - 2t)\,d\,3t + 0 = \int_0^1 (-2 + 3(2 - 2t)\,dt = (4t - 3t^2)|_0^1 = 1$$

Line Integral on C_2 by one method, parametrized by y:

The segment C_2 from $(0,3,0)$ to $(0,0,1)$ can be parametrized as $(0, y, 1 - \frac{y}{3})$ for y from 3 to 0.

Since $F(0, y, 1 - \frac{y}{3}) = (1, 1 - \frac{y}{3}, y)$, the line integral then becomes

$$\int_3^0 0 + (1 - \frac{y}{3})\,dy + y\,d(1 - \frac{y}{3}) = -\int_0^3 (1 - \frac{y}{3} - \frac{1}{3}y)dy = -(y - \frac{1}{3}y^2)|_0^3 = 0$$

Line Integral on C_3 by one method, parametrized by t:

The segment C_3 from $(0,0,1))$ to $(2,0,0)$ can be parametrized by t as $(2t, 0, (1-t))$ for t from 0 to 1.

Since $(2t, 0, (1-t)) = (1, t+1, 0)$, the line integral then becomes

$$\int_0^1 1\,d(2t) + 0 + 0 = \int_0^1 2\,dt = 2$$

Conclusion: The line integral over the entire boundary is the sum $1 + 0 + 2 = 3$. This agrees with the answer to Question 1.