Quiz 6 ANSWERS: Note: Some integration hints and a plot of the curve for Problem 2 are found on the original pdf of the Quiz, also available online.

Problem 1

For each of these two vector fields, determine whether or not the vector field is conservative. If the vector field is conservative, find a function whose gradient equals the vector field.

(a) $F(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$

Answer

Writing $F(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$, check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$:

$$\frac{\partial P}{\partial y} = e^x + \cos y = \frac{\partial Q}{\partial x}$$

Since the vector field is defined and continuous over the whole plane, it is conservative and equals ∇f for some f.

To find f, we observe that $\frac{\partial f}{\partial x} = P(x, y) = ye^x + \sin y$. Antidifferentiating with respect to x gives the indefinite integral:

$$f(x, y) = y\,e^x + x\sin y + g(y)$$

for some unknown function g that depends only on y and not on x. But we can take the partial derivative of f with respect to y and find an equation for the derivative of g:

$$Q(x, y) = \frac{\partial f}{\partial y} = e^x + x\cos y + g'(y).$$

Substituting the value of $Q(x, y)$ in this equation, we get $g' = 0$, so $g(y) = k$, for some constant k.

Thus we finally find $f(x, y) = y\,e^x + x\sin y$, or $f(x, y) = y\,e^x + x\sin y + k$ for any choice of constant k. For this f, we have $\nabla f = F$.

(b) $G(x, y) = (e^x\cos y)\mathbf{i} + (e^x\sin y)\mathbf{j}$

Answer

Writing $G(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = (e^x\cos y)\mathbf{i} + (e^x\sin y)\mathbf{j}$, check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$:

$$\frac{\partial Q}{\partial y} = -e^x\sin y\,; \quad \frac{\partial Q}{\partial x} = +e^x\sin y$$

These are not equal. Therefore, G is not conservative.

Problem 2

Consider the parametrized curve $(\cos t + \cos 2t, \sin t + \sin 2t)$, for t on the interval $[0, \pi]$. (A plot of this curve is in the pdf of Quiz 6).
Let D be the domain whose boundary consists of two parts, this curve and also the line segment from $(0, 0)$ to $(2, 0)$.

(a) Set up a line integral (ready to evaluate) that will compute the area of D.
(b) Evaluate the integral.

Answer

Let C_1 be the parametrized curve above. Let C_2 be the segment from $(0, 0)$ to $(2, 0)$. Let C be the entire boundary curve.

Then any of these line integrals will compute the area:

$$\int_C x\,dy\,; \quad \int_C -y\,dx\,; \quad \frac{1}{2}\int_C x\,dy - y\,dx$$

Here we compute the first of these: $\int_C x\,dy = \int_{C_1} x\,dy + \int_{C_2} x\,dy$.

$$\int_{C_1} x\,dy = \int_0^{\pi} (\cos t + \cos 2t)(\cos t + 2\cos 2t)dt = \frac{3}{2}\pi$$

$$\int_{C_2} x\,dy = \int_0^2 x\frac{dy}{dx}dx = \int_0^2 0\,dx = 0$$

So the area is $\frac{3}{2}\pi$.