Section 16.4 Number 22

Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{1}{2A}\int_C x^2\,dy \quad \bar{y} = -\frac{1}{2A}\int_C y^2\,dx$$

where A is the area of D.

Solution

The centroid is the same as the center of mass when the density ρ is constant. Referring to the formula on page 981, the mass m equals ρA. So the density cancels in the center of mass formula, and it becomes this formula for the centroid:

$$\bar{x} = \frac{1}{A}\int\int_D x\,dxdy \quad \bar{y} = \frac{1}{A}\int\int_D y\,dxdy$$

This can be interpreted as saying the coordinates of the centroid are the mean, or average values of x and y on D.

So we now have a double integral formula for $(\bar{x}, \bar{y})$ and a suggestion in the problem that the centroid can also be computed from line integrals on the boundary using Green's theorem. So we just plug in Green's theorem into these line integrals to convert them to double integrals and see what we get.

For the integral $\int_C Pdx + Qdy = \int_C x^2 dy$, we have

$$P(x,y) = 0; \quad Q(x,y) = x^2; \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x.$$

Then by Green's theorem

$$\frac{1}{2A}\int_C x^2\,dy = \frac{1}{2A}\int\int_D 2x\,dxdy = \frac{1}{A}\int\int_D x\,dxdy = \bar{x}$$

In the same way we find

$$-\frac{1}{2A}\int_C y^2\,dx = \frac{1}{2A}\int\int_D 2y\,dxdy = \frac{1}{A}\int\int_D y\,dxdy = \bar{y}$$

And that is all that the problem asks. Note that the area A is also a double integral that can be converted by Green's theorem to a line integral over C. But this is not something asked for by the problem. it is just assumed that one way or another the area is computed and given the name A.