

Math 309A Final Exam (Winter 2011)

NAME _____

Instructions: Show your work. You can leave radicals and fractions as they are without converting to decimals. Note that some information is given so that you do not have to compute it.

Problem 1 (35 points)

a. Find the general solution to $X'(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} X(t)$.

$$\det \begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1). \text{ Roots are } 2, -1.$$

Eigenvector for 2:

$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}. \text{ Eigenvector is } \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Eigenvector for -1:

$$\begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}. \text{ Eigenvector is } \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\text{General solution } X(t) = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

b. Initial Value Problem: Find the solution X of this equation with $X(0) = \begin{bmatrix} 15 \\ 6 \end{bmatrix}$.

$$\text{Solve } X(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

c. Find a fundamental solution matrix for this equation.

$$\begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix}$$

Problem 2 (15 points)

Check or circle the correct answer.

a) The function $f(x) = \sin(x^2)$ is

(1) odd _____ (2) even X (3) neither odd nor even.

b) If the function g is an even function defined on the interval $[-L, L]$, then the integral $\int_{-L}^L xg(x)dx$ equals

(1) zero X (2) $\frac{1}{2} \int_{-L}^L xg(x)dx$ _____ (3) cannot say without more information about the function g _____.

c) Suppose $\Phi(t)$ is a fundamental $n \times n$ solution matrix for a system $x' = Ax$, then if M is any nonsingular $n \times n$ constant matrix the following matrix is also a fundamental solution matrix for the same system:

(1) $\Phi(t)M$ X (2) $M^{-1}\Phi(t)M$ _____ (3) $M\Phi(t)$ _____

Problem 3 (10 points)

a) Consider the following differential equation for $y(t)$: $y'' - 4y' + 5y = 0$.

Find a matrix A so that if $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ is a solution of the system of differential equations $u' = Au$, then $u_1(t)$ is a solution of the differential equation for y .

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$$

b) Compute the characteristic polynomial of A .

$$\lambda^2 - 4\lambda + 5$$

Problem 4 (30 points)

- a. Find the real general solution of $X' = PX$, where $P = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

More precisely, your answer should only include real functions and real numbers the final expression that is your answer.

USE THIS INFORMATION: Eigenvalues of P are $1+2i$ and $1-2i$.

$$P - (1+2i)I = \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1-i & -1 \\ 0 & 0 \end{bmatrix} \text{ Eigenvalue is } \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \text{ or } \begin{bmatrix} 1+i \\ 2 \end{bmatrix}.$$

$$\text{Complex solution: } e^{(1+2i)t} \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = e^t \left(\cos 2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + ie^t \left(\cos 2t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$\text{Real General Solution: } c_1 e^t \left(\cos 2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 e^t \left(\cos 2t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

- b. Very briefly, sketch and/or describe the paths in the plane of the solutions of this equation and the behavior of the solutions as t approaches plus or minus infinity.

Spirals out. Origin is a source.

Problem 5 (35 points)

Consider the conduction of heat in a rod of length 40cm whose ends are maintained at 0 degrees C for all $t > 0$.

Assume that the constant α in the heat equation is 1, so the equation is

$$u_{xx}(x,t) = u_t(x,t)$$

Find a solution for the temperature $u(x, t)$, when the initial temperature in the rod is

$$u(x,0) = x, \quad 0 < x < 40$$

Let $u = XT$, then $X''T = XT'$, or $X''/X = T'/T = -k$.

Equations $X'' + kX = 0$. As usual boundary conditions $X(0) = X(40) = 0$ implies that $X(t)$ is a multiple of $\sin(n\pi t/40)$.

$$T = \exp(-(n\pi/40)^2 t).$$

Problem 6 (30 points)

Find the solution $u(x, y)$ of Laplace's equation in the semi-infinite strip

$$0 < x < \pi \text{ and } 0 < y$$

so that u satisfies the boundary conditions

$$\begin{aligned} u(0, y) = u(\pi, y) &= 0, \text{ for all } y > 0, \\ u(x, 0) &= f(x), \quad 0 \leq x \leq \pi, \end{aligned}$$

with the additional condition that $u(x, y) \rightarrow 0$ as $y \rightarrow \infty$.

Note: Since you have no formula for $f(x)$, you will not be able to compute numerical values for the coefficients for u in its series expansion, so go as far as possible and then just write a formula for the coefficients in a form ready to be evaluated if one plugs in a formula for f .

Problem 7 (20 points)

Find the eigenfunctions and eigenvalues of the given boundary value problem.

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$$

DO NOT WRITE IN THIS SPACE		
Problem #	Possible Pts	
1	35	
2	15	
3	10	
4	30	
5	35	
6	30	
7	20	
Total	175	