

Math 309A Winter 2011 MIDTERM 2

NAME (printed very legibly) _____

Problem 1 (20 points)

For some constant $L > 0$, let $f(x) = \begin{cases} -1 & \text{for } -L \leq x < 0 \\ +1 & \text{for } 0 \leq x \leq L \end{cases}$

The function $f(x)$ can be written as a Fourier series (as on the Formula Sheet).

- (a) Tell what (if any) coefficients in the Fourier series you know to be zero without computing the integral formula. Why is this true?

ANSWER: Coefficients of cosine (including the constant term) are all zero, since f is an odd function

- (b) This Fourier series for $f(x)$ converges for all real x , not just the x between $-L$ and L . What is the value that the series converges to when $x = 8L/5$? Why?

ANSWER: The series is periodic with period $2L$, so the value at $(8/5)L$ equals the value at $(8/5)L - 2L = (-3/5)L$, which is $f(-3L/5) = -1$.

- (c) Compute all the coefficients of the Fourier series for $f(x)$.

ANSWER:

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{n\pi} (1 - (-1)^n)$$

See textbook on page 598-9 for this integral. The square wave is equal to $\frac{f(x)+1}{2}$, so the series is almost the same, except for a constant and a multiple of $1/2$.

$$f(x) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi x}{L} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin \frac{(2k-1)\pi x}{L}$$

- (d) Write the first 3 non-zero terms of the series (so this will be a sum of 3 trig functions with numerical coefficients — no letters).

ANSWER: $\frac{4}{\pi} \left(\sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots \right)$

Problem 2 (20 points)

To solve the Heat Equation $\alpha^2 u_{xx} = u_t$ by the method of separation of variables, we first assume that the solution is a product $u(x,y) = X(x)T(t)$ and solve resulting equations for X and T.

Suppose that we have already found a solution $X(x) = \sin \frac{3\pi x}{5}$.

(a) Find a function T so that $u(x,y) = X(x)T(t)$ is a solution of the Heat Equation.

ANSWER:

If $u(x,y) = X(x)T(t)$ the Heat Equation becomes $\alpha^2 X''(x)T(t) = X(x)T'(t)$.

This can be rewritten as $\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = -\lambda$. Since we know what X(t) is, we can

differentiate X to compute the left-hand side as $\frac{X''(x)}{X(x)} = -\left(\frac{3\pi}{5}\right)^2 = -\lambda$

Thus the equation for T becomes

$$\frac{T'(t)}{T(t)} = -\alpha^2 \left(\frac{3\pi}{5}\right)^2 \text{ which has solution } T(t) = e^{\frac{-9\alpha^2 \pi^2 t}{25}}$$

(b) Write the solution $u(x,y) = X(x)T(t)$.

ANSWER:

$$u(x,y) = e^{\frac{-9\alpha^2 \pi^2 t}{25}} \sin \frac{3\pi x}{5}$$

(c) Usually when we solve the heat equation, we are first given boundary value conditions. What would be an example of boundary value conditions for the heat equation that would be satisfied by the function u of (b).

$$u(0,t) = u(5,t) = 0, \quad u(x,0) = \sin \frac{3\pi x}{5}$$

Also, could have $u(5/3,t) = 0$ or $u(10/3,t) = 0$ instead of $u(5,t) = 0$

Problem 3 (15 points)

(a) Write precisely what it means for a function $f(x)$ to be an even function.

ANSWER: The function is defined on a symmetric interval $[-L, L]$ or $(-L, L)$ and $f(-x) = f(x)$ for all x in this interval.

(b) Write precisely what it means for a function $f(x)$ to be an odd function.

ANSWER: The function is defined on a symmetric interval $[-L, L]$ or $(-L, L)$ and $f(-x) = -f(x)$ for all x in this interval.

Circle the correct answer:

(c) The function $e^{2t} - e^{-2t}$ is (i) an even function (2) an odd function (3) neither even nor odd.

ANSWER: (2) an odd function

(d) The function $e^{2t} \sin t$ is (i) an even function (2) an odd function (3) neither even nor odd.

ANSWER: (3) neither even nor odd.

(e) The function $f(x) = -3$ is (i) an even function (2) an odd function (3) neither even nor odd.

ANSWER: (i) an even function

(f) The product of two odd functions is (i) always an even function (2) always an odd function (3) not always either (1) nor (2).

ANSWER: (i) always an even function

(g) The sum of an odd function and an even function is (i) always an even function (2) always an odd function (3) not always either (1) nor (2).

ANSWER: (3) not always either (1) nor (2).

Problem 4 (30 points)

(a) Find the general solution of the differential equation $x'(t) = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} x(t)$.

ANSWER: The characteristic polynomial is $\det \begin{bmatrix} -\lambda & 1 \\ -4 & 4 - \lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

So the eigenvalues are 2 and 2 (double roots). Solve for eigenvector to get one

solution: $(A - 2I)\mu = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \mu = 0$ has solution $\mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so DE has first solution

$X_1(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. For second solution, we try $X_2(t) = te^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t} \zeta$

Substitute in the DE and solve for unknown vector.

$$X_2'(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2te^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2e^{2t} \zeta = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} te^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} e^{2t} \zeta = 2te^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} e^{2t} \zeta$$

$$\text{Rearrange to get } e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) e^{2t} \zeta = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} e^{2t} \zeta$$

So solve $\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \zeta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to get solution. There are an infinite number of solutions.

One is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and another is $\begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$. Each gives a solution to the DE. We will choose

the first to give $X_2(t) = te^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The general solution is $c_1 X_1(t) + c_2 X_2(t)$.

(b) Find a fundamental solution matrix $\Psi(t)$ for this differential equation.

ANSWER: Put the two independent solutions as columns into a matrix.

$$\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (1+2t)e^{2t} \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & t \\ 2 & 1+2t \end{bmatrix}$$

(c) Find the fundamental solution matrix $\Phi(t)$ for which $\Phi(0) = I$.

ANSWER: $\Theta(t) = \Psi(t)\Psi(0)^{-1}$. Since $\Psi(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ then $\Psi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and

$$\Theta(t) = \Psi(t) \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & t \\ 2 & 1+2t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 1-2t & t \\ -4t & 1+2t \end{bmatrix}$$

(d) Find one solution for $x'(t) = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} x(t) + \begin{bmatrix} e^t \\ 1 \end{bmatrix}$

ANSWER: by Method of Undetermined Coefficients. Break into two equations.

Denote by A the matrix $\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$

Solve $Y_1'(t) = AY_1(t) + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$ by setting $Y_1(t) = e^t \mu$, substituting to get equation for

unknown vector: $(A - I)\mu = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Solve for μ ; then DE has solution $Y_1(t) = e^t \begin{bmatrix} -3 \\ -4 \end{bmatrix}$.

Solve $Y_2'(t) = AY_2(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by setting $Y_2(t) = v$, substituting to get equation for

unknown vector: $(A - 0I)v = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Solve for v ; then DE has solution $Y_2(t) = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$.

(Details are in your class notes for March 9.)

(e) Write the general solution for the equation of (d).

ANSWER:

$$x(t) = e^t \begin{bmatrix} -3 \\ -4 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} + c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left(t e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Problem 5 (15 points)

Given differential equation $y''(t) + \lambda y(t) = 0$, with boundary condition $y'(0) = y'(4) = 0$.

Find the eigenvalues and eigenfunctions of this boundary value problem.

In other words:

- (a) What are the values of λ for which there is a non-zero solution of this boundary value problem?

ANSWER:

If $\lambda < 0$, then $\lambda = -a^2$ and the equation has solutions $y(t) = c_1 \cosh at + c_2 \sinh at$ and $y'(t) = ac_1 \sinh at + ac_2 \cosh at$

The condition $y'(0) = 0$ implies that $c_2 = 0$. But then the condition that $y'(4) = 0$ is only true if $c_1 = 0$. So there are no nonzero solutions for this case.

If $\lambda = 0$, then $y(t) = c_1 t + c_2$ and $y'(t) = c_1$ so the boundary condition is satisfied if $c_1 = 0$ and $y(t) = c_2$.

If $\lambda > 0$, then $\lambda = a^2$ and the equation has solutions $y(t) = c_1 \cos at + c_2 \sin at$ and $y'(t) = -ac_1 \sin at + ac_2 \cos at$.

The condition $y'(0) = 0$ implies that $c_2 = 0$. But then the condition that $y'(4) = 0$ is only true if $c_1 = 0$ or $\sin 4a = 0$. So the nonzero solutions for this case require that $4a$ be an integer multiple of π .

So there are non-zero solutions for $\lambda = \left(\frac{n\pi}{4}\right)^2$, for nonnegative integers n .

- (b) For each eigenvalue, tell the solutions $y(t)$ of the boundary value problem.

ANSWER:

$y(t) = \cos \frac{n\pi t}{4}$ for nonnegative integers $n = 0, 1, 2, 3, \dots$