

Math 309A Midterm 1 (Winter 2011)

NAME _____

Instructions: Show your work. You can leave radicals and fractions as they are without converting to decimals. Note that some information is given so that you do not have to compute it.

Problem 1 (13 Points)

- a. Compute the length of the complex vector $\begin{bmatrix} 3-4i \\ 1+i \end{bmatrix}$

ANSWER: $\begin{bmatrix} \overline{3-4i} & \overline{1+i} \end{bmatrix} \begin{bmatrix} 3-4i \\ 1+i \end{bmatrix} = 9+16+1+1 = 27$

So the length = $\sqrt{27}$.

- b. Let $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Compute the determinant of $M - \lambda I$ and find the eigenvalues of M but **NOT THE EIGENVECTORS**.

ANSWER: $\det(M - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = \lambda^2(2-\lambda) - (2-\lambda) = (\lambda^2 - 1)(2-\lambda)$

So the eigenvalues are 2, +1, -1

Problem 2 (26 points)

- a. Find the general real solution to the differential equation $X'(t) = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} X(t)$.

ANSWER: Characteristic equation is $\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$. So eigenvalues = 5 and -1.

Eigenvector for 5 is in null space of $\begin{bmatrix} 3-5 & 2 \\ 4 & 1-5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, so an eigenvector for 5 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvector for -1 is in null space of $\begin{bmatrix} 3+1 & 2 \\ 4 & 1+1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, so an eigenvector for -1 is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. So

GENERAL SOLUTION: $X = c_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

- b. Initial Value Problem: Find the solution X of this equation with $X(0) = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$.

ANSWER: Solve $\begin{bmatrix} c_1 + c_2 \\ c_1 - 2c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and get $X = \frac{1}{3} e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{20}{3} e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

- c. Find a fundamental matrix for this equation.

ANSWER: One of the fundamental matrices is $\begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -2e^{-t} \end{bmatrix}$

- d. Compute the Wronskian of this fundamental matrix.

ANSWER: Wronskian = $\det \begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -2e^{-t} \end{bmatrix} = -2e^{4t} - e^{4t} = -3e^{4t}$

Problem 3 (26 points) In answering this question, *be sure to notice* the information that you are given that *you do not have to compute*.

Let matrix $B = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. The characteristic polynomial $\det(B - \lambda I) = -\lambda(\lambda^2 + 4)$.

a. Find the general real solution of $X' = BX$.

ANSWER: Eigenvalues are 0, +2i, -2i.

$$\text{Eigenvector for } 0: B - 0I = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Eigenvector} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\text{Eigenvector for } 2i: B - 2iI = \begin{bmatrix} -2i & 2 & -2 \\ 0 & -2i & 0 \\ 2 & 0 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Eigenvector} = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}.$$

Take real and imaginary parts of

$$e^{2it} \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} = (\cos 2t + i \sin 2t) \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} i \cos 2t - \sin 2t \\ 0 \\ \cos 2t + i \sin 2t \end{bmatrix} = \begin{bmatrix} -\sin 2t \\ 0 \\ \cos 2t \end{bmatrix} + i \begin{bmatrix} \cos 2t \\ 0 \\ \sin 2t \end{bmatrix}$$

General real solution:

$$x(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\sin 2t \\ 0 \\ \cos 2t \end{bmatrix} + c_3 \begin{bmatrix} \cos 2t \\ 0 \\ \sin 2t \end{bmatrix}$$

b. Initial Value Problem: Find the solution X of this equation with $X(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$$\text{ANSWER: } x(t) = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -\sin 2t \\ 0 \\ \cos 2t \end{bmatrix} + 1 \begin{bmatrix} \cos 2t \\ 0 \\ \sin 2t \end{bmatrix}$$

c. Describe the behavior of this solution as t increases towards infinity.

ANSWER: Circles the origin.

d. Find a real fundamental solution matrix for this differential equation.

ANSWER: Matrix whose columns are the 3 vector functions in the general solution.

Problem 4 (35 points)

- a. Find the general real solution of $X' = PX$, where $P = \begin{bmatrix} -1 & 9 \\ -1 & -1 \end{bmatrix}$.

ANSWER: Equation $\det(P - \lambda I) = \det \begin{bmatrix} -1 - \lambda & 9 \\ -1 & -1 - \lambda \end{bmatrix} = (-1 - \lambda)^2 + 9 = 0$ has solutions $\lambda = -1 \pm 3i$.

Eigenvector for $-1+3i$: $\begin{bmatrix} -1 - (-1+3i) & 9 \\ -1 & -1 - (-1+3i) \end{bmatrix} = \begin{bmatrix} -3i & 9 \\ -1 & -3i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3i \\ 0 & 0 \end{bmatrix}$ so an

eigenvector is $\begin{bmatrix} 3i \\ -1 \end{bmatrix}$ and a complex solution is

$$e^{(-1+3i)t} \begin{bmatrix} 3i \\ -1 \end{bmatrix} = e^{-t} \begin{bmatrix} 3i(\cos 3t + i \sin 3t) \\ -\cos 3t - i \sin 3t \end{bmatrix} = -e^{-t} \begin{bmatrix} 3 \sin 3t \\ \cos 3t \end{bmatrix} + i e^{-t} \begin{bmatrix} 3 \cos 3t \\ -\sin 3t \end{bmatrix}$$

So the general real solution is this (the coefficients are real, and a -1 was absorbed into the first constant):

$$X(t) = c_1 e^{-t} \begin{bmatrix} 3 \sin 3t \\ \cos 3t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 3 \cos 3t \\ -\sin 3t \end{bmatrix}$$

- b. Initial Value Problem: Find the solution X of this equation with $X(0) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

ANSWER: Solve $X(0) = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ to get $X(t) = 3e^{-t} \begin{bmatrix} 3 \sin 3t \\ \cos 3t \end{bmatrix} + 2e^{-t} \begin{bmatrix} 3 \cos 3t \\ -\sin 3t \end{bmatrix}$.

- c. Describe how this solution behaves as t increases to towards positive infinity.

ANSWER: Spirals inward towards the origin.

- d. Compute the fundamental matrix e^{tP} .

ANSWER: The columns of the matrix are the solutions of the initial value problem for values $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The columns are $\frac{1}{3}e^{-t} \begin{bmatrix} 3 \cos 3t \\ -\sin 3t \end{bmatrix}, 1e^{-t} \begin{bmatrix} 3 \sin 3t \\ \cos 3t \end{bmatrix}$, so the matrix is

$$\begin{bmatrix} e^{-t} \cos 3t & 3e^{-t} \sin 3t \\ -\frac{e^{-t}}{3} \sin 3t & e^{-t} \cos 3t \end{bmatrix}.$$