Midterm – Math 308 – Winter 2002

Throughout the exam, recall that e_i denotes the vector with coefficients all zero except in the i^{th} coordinate, which is a 1.

Problem 1: Consider the system of equations

$$\begin{array}{rrrrl} & -2x_2 & +6x_3 & +2x_4 & =-1 \\ x_1 & +3x_2 & +3x_3 & -2x_4 & = 3 \\ x_1 & +4x_2 & & & =-1 \end{array}$$

- Write down the ***augmented matrix*** of this system of equations.

ANSWER:

$$\begin{bmatrix} 0 & -2 & 6 & 2 & -1 \\ 1 & 3 & 3 & -2 & 3 \\ 1 & 4 & 0 & 0 & -1 \end{bmatrix}$$

- Use elementary row operations to find a matrix in ***reduced echelon form*** (REF) which is row equivalent to the matrix you found in (a). **DO NOT SOLVE THE EQUATION!**

ANSWER: Note that there is only one REF form. Just an echelon form is not a complete answer to the question.

$$\textbf{REF} = \begin{pmatrix} 1 & 0 & 12 & 0 & 3 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -3/2 \end{pmatrix}$$

Problem 2:
Suppose the augmented matrix of a different matrix equation $Bx = u$ reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

· Find the ***general solution*** of Bx = u and write the answer in ***vector parametric form***.

ANSWER: $$x = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

· Is the matrix function Bx one-to-one? Yes ___ No X_ .
An infinite number of vectors x have same product Bx = u, as shown in (a), since there are 2 free variables.

· Is the matrix function Bx onto? Yes _X__ No ___.
The equation Bx= u is consistent for all u, since there is a pivot in each row of B.

Problem 3: Let T be the linear transformation from R^2 to R^3 such that

$T(e_1) = 2e_1 - e_2 + 2e_3$
$T(e_2) = e_1 + e_2 + e_3$

(a) Find the ***value*** of $T(3e_1 + 2e_2)$

ANSWER: $$T(3e_1 + 2e_2) = 3\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 7 \end{bmatrix}$$

(b) What is the ***standard matrix*** for T?

ANSWER: $$\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$$

Problem 4: Consider the quadratic polynomial $y = p(t)$ such that the graph of $p(t)$ goes through the points (-1,2), (1,5), (2,6) in the (t,y) plane. Set up the matrix equation which would allow you to determine the polynomial $p(t)$, **but don't solve it**. Make sure you label things clearly so that we can tell *why* your equation would help you determine $p(t)$.

ANSWER: *If* $p(t) = at^2 + bt + c$, *then the coefficients of the polynomial satisfy*

$$\begin{aligned} 1a + (-1)b + c &= 2 \\ a + b + c &= 5 \\ 4a + 2b + c &= 6 \end{aligned}$$

In matrix form Ax=b:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

(If you set $p(t) = a + bt + ct^2$ *then the columns of A appear in a different order.) Compare this with* ***Section 1.2 #35*** *from Assignment 1.*

Problem 5: Write True or False.

__F__If a system of linear equations has 4 equations in 5 unknowns, then it must have at least one solution.
The system can still be inconsistent. If it is consistent such a system has an infinite number of solutions.

__F__A is a 3x2 matrix with 2 pivot positions. Ax = 0 has a nontrivial solution.
The solution to this equation has no free variables, since there is a pivot in every column.

__F__A is a 3x2 matrix with 2 pivot positions. Ax = b has at least one solution for every possible b. *There is not a pivot in every row, so the REF of A has a zero row. Thus for some b the equation is inconsistent.*

Problem 6: Complete the definition of *linearly independent*:

A set of vectors {a_1, a_1, … , a_n} is linearly independent if *none of the vectors is a linear combination of the others.*

Alternate Definition: A set of vectors {a_1, a_1, … , a_n} is linearly independent if *the equation $Ax = 0$ has only the trivial solution, where A is the matrix with columns [a_1 a_2 … a_n].*

Note: If your definition talks about solutions and there is no equation stated in the definition, it is not self-contained. If your reader had not attended 308, it would be impossible to decipher. Look up the official definition in the Glossary or in the text.

Problem 7:

A is a matrix with columns a_1, a_2, etc. If $Ax = y$ has general solution $\begin{pmatrix}1\\2\\3\\0\\5\end{pmatrix} + x_4\begin{pmatrix}2\\-1\\2\\1\\0\end{pmatrix}$,

then:

a) Write y as a linear combination of the columns of A, using numbers for coefficients (no variables).

ANSWER: *The vector* $x = \begin{pmatrix}1\\2\\3\\0\\5\end{pmatrix}$ *is one solution to* $Ax = y$ *(set* $x_4 = 0$ *to get this).*

Thus $a_1 + 2a_2 + 3a_3 + 5a_5 = y$.

There are an infinite number of other solutions obtained by setting x_4 *equal to other values such as 1, 2, or any other number. NOTE: Compare this with* ***Section 1.4, Practice Problem 1 on page 45.***

b) Write one example of a nontrivial solution of $Ax = 0$ if there is one. Example should only have numbers, no letters.

ANSWER: *Choose* $x = \begin{pmatrix}2\\-1\\2\\1\\0\end{pmatrix}$ *or any other scalar multiple of this vector. NOTE: Read pp. 51-52.*

c) Are the columns of A linearly dependent? Check one:

Yes. ___X____ The columns are definitely dependent.

Because we saw in (b) that there is a nontrivial solution to $Ax = 0$.

No. ________ The columns are definitely linearly independent.

Maybe. _____ There is not enough information given about A to be sure.