How to write a set of eigen-equations as one matrix equation

Explanation

Suppose that we have a matrix A and a set of vectors $S_1, S_2, \ldots, S_k$ and a set of numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $AS_i = \lambda_i S_i$ for each *i*. In other words, each S_i is an eigenvector of A for eigenvalue λ_i.

Now we use one of our important formulations of matrix product. In general, if M is a matrix with these columns: $M = [M_1\ M_2 \ldots M_k]$, then $AM = [AM_1\ AM_2 \ldots AM_k]$. We call this "***column-wise left matrix multiplication***".

Thus if we form a matrix $S= [S_1\ S_2 \ldots S_k]$, with the eigenvectors as columns, then $AS = [AS_1\ AS_2 \ldots AS_k]$, so $AS = [\lambda_1 S_1\ \lambda_2 S_2 \ldots \lambda_k S_k]$ for these eigenvectors.

Now again using *column-wise left matrix multiplication* applied to S as the matrix on the left, we see that $Se_1 = S_1$, $Se_2, = S_2$, etc., where e_i is the vector with entry 1 in the *i*-th row and zero elsewhere. So also $S(\lambda_i e_i) = \lambda_i S_i$ for each *i*. In this case the vector $\lambda_i e_i$ is the vector with entry λ_i in the *i*-th row and zero elsewhere.

If we form a matrix D with these columns $\lambda_i e_i$ we get $D = [\lambda_1 e_1\ \lambda_2 e_2 \ldots \lambda_k e_k]$. D is a ***diagonal matrix*** with the eigenvalues down the main diagonal.

Putting this together, we get $AS = [\lambda_1 S_1\ \lambda_2 S_2 \ldots \lambda_k S_k] = [S\lambda_1 e_1\ S\lambda_2 e_2 \ldots S\lambda_k e_k] = SD$.

Example

Let $A = \begin{bmatrix} 1 & 6 \\ 3 & -2 \end{bmatrix}$. If $S_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $S_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then $AS_1 = 4S_1$ and $AS_2 = (-5)S_2$.

So $\boxed{AS = \begin{bmatrix} 1 & 6 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix} = SD}$.

Further Equations: Similarity

In the case when A is an nxn matrix, if k=n and the eigenvectors form a basis for Rn, then the matrix S is nonsingular and can be inverted. This leads to two further equations:

$$AS = SD \Rightarrow S^{-1}AS = S^{-1}SD = D \Rightarrow D = S^{-1}AS$$

$$AS = SD \Rightarrow ASS^{-1} = SDS^{-1} \Rightarrow A = SAS^{-1}$$

We will see in the section on ***similarity*** that this means that with a change of coordinate, the matrix function given by A looks like a diagonal matrix function. This means that powers of A and other computations can be simplified.