Math 308 Midterm 2

ANSWERS IN RED

1A Spaces from a Matrix

Let $A = \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 6 \\ 2 & 0 & 0 & 4 & 8 \end{bmatrix}$. A reduces: $A \to \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Find a basis for each of the following spaces. For full credit BOX and label your answers clearly.

a. Range of A

One basis is obtained by choosing the columns 1 and 3 corresponding to pivots (steps):

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

A second option would have been to reduce A^T to echelon form an take the non-zero rows (written in column form) as a basis.

b. Null Space of A

From the echelon form: x_2, x_4, x_5 are free and

$x3 = -3x_5$

$x1 = x_3 -2\, x_4 - x_5 = -2\, x_4 - 4\, x_5$

(Notice the substitution to remove the dependent variable x_3 from the right side!)

Option: To avoid the substitution we could have reduced A further to the RREF =

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the solution $X = \begin{bmatrix} -2x_4 - 4x_5 \\ x_2 \\ -3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

And a basis for N(A) is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

c. Range of A^T

One basis for range of A^T is the set of non-zero rows of the RREF of A (transposed to column form). *THERE ARE MANY CORRECT ANSWERS TO THIS PROBLEM BUT ALL OF THEM CONSIST OF TWO VECTORS IN R^5.*

$$\left\{\begin{bmatrix}1\\0\\0\\2\\4\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\3\end{bmatrix}\right\}$$

d. Null Space of A^T

One method: Row reduce A^T. Solve $A^TX = 0$.

$$A^T = \begin{bmatrix}1 & 0 & 2\\0 & 0 & 0\\-1 & 2 & 0\\2 & 0 & 4\\1 & 6 & 8\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 2\\0 & 0 & 0\\0 & 2 & 2\\0 & 0 & 0\\0 & 6 & 6\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Solution is $X = \begin{bmatrix}-2x_3\\-x_3\\x_3\end{bmatrix} = x_3\begin{bmatrix}-2\\-1\\1\end{bmatrix}$. Basis is $\left\{\begin{bmatrix}-2\\-1\\1\end{bmatrix}\right\}$.

1B Matrix Relationships

For the matrix A in the previous problem, write down

a. Rank of A _________Rank of A = 2________

b. Nullity of A ________ Nullity of A = 3_________

c. Rank of A^T_______ Rank of A^T = 2________

d. Nullity of A^T______ Nullity of A^T = 1__________

Write down 3 equations expressing relationships among the 4 quantities of rank and nullity of M and M^T that hold for ANY m x n matrix M.

i. Rank M + Nullity M = n

ii. Rank M^T + Nullity M^T = m

iii. Rank M = Rank M^T

Then verify that these relationships hold for your answers to a, b, c, d in this problem.

i. Rank M + Nullity M = 2 + 3 = 5

ii. Rank M^T + Nullity M^T = 2 + 1 = 3

iii. Rank M = 2 = Rank M^T

2. Linear Fit

Find the least squares linear fit to the given data.

t	-1	0	1
y	-1	2	2

Your answer should be the equation of a line. Box your answer.

The equation of a line is y = mx + b. The 3 data points give 3 equations:

i. -m + b = -1

ii. 0m + b = 2

iii. m + b = 2

It is easy to see that there is no solution to this. You can go the augmented matrix route

or just notice by (ii) that b = 2 and so by (i) m = -3 and b (iii) m = 0! So by least squares,

we write this as $A\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ with $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

To solve by least squares, we multiply both sides by A^T and solve.

$$A^T A\begin{bmatrix} m \\ b \end{bmatrix} = A^T \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$A^T A\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} \text{ and } A^T\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \text{ so}$$

$$A^T A\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$, so m = 3/2 and b = 1.

ANSWER: The line is y = (3/2)x + 1

3. Orthogonal Basis

(a) Find an orthogonal basis for W, where W is the span of $\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}\right\}$.

Let this set of vectors be denoted by $\{w_1, w_2\}$. Then the component v of w_2 in the direction w_1 is given by the projection formula: $v = aw_1$, where $a = w_2^T w_1 / w_1^T w_1$,

Thus a = (0 + 0 + 0 + 10)/(1 + 1 + 4 + 4) = 10/10 = 1, so $v = 1\ w_1$.

For an orthogonal basis {u1, u2} we set

$$u_1 = w_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \text{ and } u_2 = w_2 - au_1 = w_2 - u_1 = \begin{bmatrix} -1 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

(b) Find an orthonormal basis for W.

Take the same vectors and divide by their lengths.

$$u_1^T u_1 = |u_1|^2 = 1 + 1 + 4 + 4 = 10; \quad |u_1| = \sqrt{10}$$

$u_2^T u_2 = |u_2|^2 = 1+1+4+9 = 15; \quad |u_2| = \sqrt{15}$

An orthonormal basis is $\left\{ \left(\frac{1}{\sqrt{10}}\right)\begin{bmatrix}1\\1\\2\\2\end{bmatrix}, \ \left(\frac{1}{\sqrt{15}}\right)\begin{bmatrix}-1\\-1\\-2\\3\end{bmatrix} \right\}$

4. Linear Transformations

(a) For X in R^3 define $T(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}) = (x_1 - x_2 + 2x_3)\begin{bmatrix}1\\3\end{bmatrix}$.

Is T a linear transformation? Yes _X_ No ____

If yes, what is the matrix of T?

Expand $T(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}) = (x_1 - x_2 + 2x_3)\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}x_1 - x_2 + 2x_3\\3x_1 - 3x_2 + 6x_3\end{bmatrix} = \begin{bmatrix}1 & -1 & 2\\3 & -3 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

So matrix is $\begin{bmatrix}1 & -1 & 2\\3 & -3 & 6\end{bmatrix}$

(b) For X in R^2 define $S\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1+2\\1+x_2\end{bmatrix}$.

Is S a linear transformation? Yes ____ No X_ (many ways to show this: simplest is S(0) is not 0.

If yes, what is the matrix of S?

(c) Let P(X) be the rotation of a point X in R^2 counterclockwise around center of rotation 0 by an angle of 90 degrees. What is the matrix of P?

Matrix is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ since this rotation moves e1 to e2 (first column) and e2 to -e1 (second column. Also this is a special case of the general rotation matrix with trig functions in it.

(d) U is a linear transformation from R^2 to R^2 so that

$$U\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\0\end{bmatrix} \text{ and } U\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}.$$

What is the matrix of U?

Method 1: These two vector equations can be combined into one matrix equation:

$$U\begin{bmatrix}1&0\\1&1\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix} \text{ so } U=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}=\begin{bmatrix}2&0\\-1&1\end{bmatrix}$$

Method 2: The columns of the matrix of U are $U(e_1)$ and $U(e_2)$. So we need to compute

these values. Solve $\begin{bmatrix}1\\0\end{bmatrix}=e_1=c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$. Solution is $\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$.

So $U(e_1)=U\left(1\begin{bmatrix}1\\1\end{bmatrix}+(-1)\begin{bmatrix}0\\1\end{bmatrix}\right)=1U\left(\begin{bmatrix}1\\1\end{bmatrix}\right)+(-1)U\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}2\\0\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$

In the same way we write e2 as a combination of the same vectors and find $U(e_2)=\begin{bmatrix}0\\1\end{bmatrix}$

5. Short Answer

(a) Let A be a 3 x 15 matrix.

What are the possible values of the rank of A?

Possible ranks are 0, 1, 2, 3 (can't be more than the number or rows or the number of columns)

What are the possible values of the nullity of A?

Possible nullity: 15, 14, 13, 12

(b) True or False. If B is a matrix with nullity = 0, the columns of B are linearly independent. True or False __True____

Why? Nullity = 0 (the number) means null space is {0} (the vector) so the only solutions of Bx = 0 is x = 0. Since $Bx = x_1B_1 + \ldots x_nB_n$, this says the columns are linearly independent directly from the definition. (Note: this answer is MUCH longer than a student answer should be. I keep trying to explain. ☺)

(c]) Let $v_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$. Suppose that X is a vector in R^3 and that we know the dot products of X with these vectors, namely:

$X^T v_1 = 2, X^T v_2 = 2, X^T v_3 = 2.$

What is the vector X? (This should be an answer X = some vector with only numbers in it, not letters. Hint: There may be some orthogonal vectors here somewhere.)

(i) ***Essential***: *Check that v1, v2, v3 is an orthogonal set* by taking all three dot products of pairs of these vectors and noting the products are all equal to zero.

(ii) Use the formula for coordinates with respect to an orthogonal basis.

$$X = \left(X^T v_1 / v_1^T v_1\right) v_1 + \left(X^T v_2 / v_2^T v_2\right) v_2 + \left(X^T v_3 / v_3^T v_3\right) v_3 = (2/8)v_1 + (2/2)v_2 + (2/9)v_3$$

This is actually an OK answer as it stands, but if you want numbers, then just compute:

$$X = (2/8)v_1 + (2/2)v_2 + (2/9)v_3 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 2/3 \\ 3/2 \end{bmatrix}$$