Problem 1: Solving a linear equation

Given matrix $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -3 & -7 & 0 & 0 \\ -1 & -4 & 0 & 10 \end{bmatrix}$ and vector $y = \begin{bmatrix} 5 \\ -8 \\ 9 \end{bmatrix}$.

(a) Solve Ax = y (if the equation is consistent) and write the general solution x in (vector) parametric form.

$$x = \begin{bmatrix} 19 \\ -7 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -14 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

Write a basis for the null space of A. **Basis =** $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -14 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) What is the dimension of the range of A? **Dimension = 2**

(c) Is y in the span of the row vectors of A? **No. Not even in $\mathbf{R}^4$.**

Problem 2: Conclusions from echelon form.

In each case, we start with a matrix A and vector and tell what one will get by reducing the augmented matrix of the system Ax = y to echelon form. Answer the questions in each case using this information. **It is possible some equations will have no solution**.

A	**y**	**Echelon form of augmented matrix of Ax = y.**
$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 3 & 2 \\ 4 & 5 & 4 & 4 \end{bmatrix}$	$y = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 & 0 & -5 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

(a) Write the general solution for Ax = y in (vector) parametric form

Solution: $x = \begin{bmatrix} -5 \\ 3 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(b) What is the dimension of the null space of A? **Dimension =1**

(c) Write down a basis for the null space of A. **Basis =** $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(d) Is y in the range of A? **Yes**

(e) What is the dimension of the range of A? **Dimension = 3. Note Range(A) =** $\mathbf{R}^3$ **!**

(f) Write down a basis of the range of A. **Basis =** $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$ **, or** $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ **by (e).**

(g) Are the rows of A independent? **Yes**

(h) What is the dimension of the row space of A? **Dimension = 3**

B	**z**	**Echelon form of augmented matrix of Bx = z.**
$B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 8 \end{bmatrix}$	$z = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(a) Write the general solution for Bx = z in (vector) parametric form
Solution: NO SOLUTION. Inconsistent system.

(b) What is the dimension of the null space of B? **Dimension = 0**

(c) Write down a basis for the null space of B. **Basis = {} empty basis**

(d) Is z in the range of B? **No**

(e) What is the dimension of the range of B? **Dimension =** 2

(f) Write down a basis of the range of B. **Basis = the two column vectors of B**

(g) Are the columns of B independent? **Yes**

C	**Reduced row echelon form of C.**
$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(a) Is C invertible? **No**
(b) Are the columns of C independent? **No**
(c) Write down a basis for the null space of C. $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

Problem 3: Compute matrix products and inverses

Compute the stated matrix products (if defined) for these matrices.

$$A = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 & 0 & -1 \end{bmatrix}$$

Compute each of the following matrix products or other matrices (if defined):

$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}$	C^{-1} No inverse (not square)
$AB = \begin{bmatrix} 5 & 1 & 9 & 8 \\ 2 & 1 & 3 & 5 \end{bmatrix}$	BA Not defined.
$CD = \begin{bmatrix} 2 & 4 & 0 & -2 \\ 1 & 2 & 0 & -1 \\ 3 & 6 & 0 & -3 \\ 1 & 2 & 0 & 1 \end{bmatrix}$	$BC = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$
CD^T not Defined	C^TD not defined

Problem 4: Find the eigenvalues and eigenvectors

Find the eigenvalues and eigenvectors of matrix $F = \begin{bmatrix} 0 & 5 \\ -5 & 10 \end{bmatrix}$.

Eigenvalues = 5, 5. There is only one eigenvalue.

Eigenspace basis = .$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ If possible, diagonalize F. **NOT POSSIBLE.**

Problem 5: Given the eigenvalues find the eigenvectors

Given that **1 and -3 are the eigenvalues** of the matrix $C = \begin{bmatrix} -3 & 8 & 2 \\ -1 & 9 & 2 \\ 4 & -8 & -1 \end{bmatrix}$, find the **eigenvectors** of this matrix. **Hint**: First find whether there is a third eigenvalue. If you compute the determinant of C, knowing 2 eigenvalues, you can find the third root of the characteristic polynomial without computing det(C - λI).

det(C) = -21, so the third eigenvalue k = -21/-3 = 7

Eigenvalue	Eigenvector
1	$\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$
-3	$\begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix}$
7	$\begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

If possible, diagonalize C, i.e., write C = MDN, where D is diagonal. You **DO NOT** need to compute the inverse of any matrix in this problem. If a matrix is the inverse of a known matrix, just write it as the inverse

M = $\begin{bmatrix} 0 & -4 & -2 \\ -1 & -1 & -3 \\ 4 & 4 & 2 \end{bmatrix}$

D = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

N = $\mathbf{M^{-1}}$

Problem 6: Compute orthogonal projections

Let $h = \begin{bmatrix} 6 \\ 12 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$

(a) Compute m = **the projection of h on Span({u}).** (The formula should be computed numerically, but you need not simplify fractions, etc., in your answer.)

$$m = \frac{21}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$$

(b) Compute g = **the projection of h on Span({u, v}).** (The formula should be computed numerically, but you need not simplify fractions, etc., in your answer.)

$g = \frac{21}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{12}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 3 \end{bmatrix}$ **by applying the projection formula with dot product to an orthogonal basis for span {u, v} obtained by Gram-Schmidt.**

Another method would use the normal equation $A^TAx = A^Th$:

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 12 \\ 3 \end{bmatrix}$, that is this, after multiplying $\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} x = \begin{bmatrix} 21 \\ 18 \end{bmatrix}$

The solution to this is $x^* = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, so $g = Ax^* = \begin{bmatrix} 9 \\ 9 \\ 3 \end{bmatrix}$

Problem 7: Matrix of Linear Transformation

(a) If T is the linear transformation of R^2 that rotates the plane by an angle so that the point (1, 0) is rotated to (3/5, 4/5). What is the matrix A of this transformation?

$$\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} = A$$

(b) Is the matrix A an orthogonal matrix? **Yes**

(c) Is the matrix 2A an orthogonal matrix? **No**

Show why in both cases. **Check definition $A^TA = I$**

(d) Given the vector $u = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, if x is a vector in R^3, then let T(x) = $\left(u^T x\right)u$. Is T a linear transformation? **YES**

If so, what is its matrix (with respect to the standard basis)?

$\begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & -3 \\ 3 & -3 & 9 \end{bmatrix}$. This is = uu^T.

Problem 8: Least squares solution

You have the following data for variables x, y, z

x	y	z
0	0	3
1	0	4
0	1	4
1	1	1

(a) Now suppose that you want to fit this data to a function z = a + bx + cy, where, a, b, c are some constants that you need to solve for. Write a system of equations for variables **a, b, c** that can be solved if there is an exact fit of the function to the data.

a + 0b + 0c = 3
a + b + 0c = 4
a + 0b + c = 4
a + b + c = 1

Or
a = 3
a + b = 4
a + c = 4
a + b + c = 1

Note: by inspection, it is easy to see that a solution of the first three equations is a = 3, and b = c = 1 but then a + b + c = 5. So this system is inconsistent.

(b) Find the lease-squares "solution" a, b, c to this system and check the values of z = a + bx + cy at the data points to see how close a fit it is.

Solve $A^TAx = A^Tt$:

$$\begin{bmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\\1&1&1\end{bmatrix}x=\begin{bmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\end{bmatrix}\begin{bmatrix}3\\4\\4\\1\end{bmatrix}$$

, which multiplies out to

$$\begin{bmatrix}4&2&2\\2&2&1\\2&1&2\end{bmatrix}x=\begin{bmatrix}12\\5\\5\end{bmatrix}$$

. The solution to this equation is $x=\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}4\\-1\\-1\end{bmatrix}$**.**

Least squares solution to original problem: z = 4 – x – y.