PROBLEM 57 OF 1.5

In this problem we are given a matrix P The assertion in the text is that if X is a vector whose entries are the 3 subscription numbers for Newspaper A, Newspaper B, and None, then the product PX will be the vector of subscription numbers after one year. So if we do the computation, for the suggested value of X, we get

$$PX = \begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix} \begin{bmatrix} 150000 \\ 100000 \\ 50000 \end{bmatrix} = \begin{bmatrix} 135000 \\ 120000 \\ 45000 \end{bmatrix}$$

Then this is the new vector of subscription numbers. So to find out what the subscription numbers will be after another year, we multiply this vector by P:

$$P(PX) = \begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix} \begin{bmatrix} 135000 \\ 120000 \\ 40000 \end{bmatrix} = \begin{bmatrix} 126000 \\ 132000 \\ 40000 \end{bmatrix}$$

1. Why do we multiply X by matrix P?

So this is the answer to the Homework Problem, but it leaves open some important questions that we should answer in order to understand this example and this kind of application of matrices.

Question: Why do we multiply X by this matrix P to find the subscription vector after a year?

Answer: What we are told, is the the first column represents what happens to the subscribers of Newspaper A, that 70% remain with the paper, 20% switch to Newspaper B and the remaining 10% will change to No Subscription to A or B. So if we multiply these percentages by the number of subscribers to A, we will get a vector with 3 numbers representing the Subscribers to A, to B and to None after a year. Since the initial number of subscribers to A is 150,000, this vector is

$$150000 \begin{bmatrix} .70 \\ .20 \\ .10 \end{bmatrix}$$

Likewise, the corresponding vectors for Newspaper B and for None will be

$$100000 \begin{bmatrix} .15 \\ .80 \\ .05 \end{bmatrix} \text{ and } 50000 \begin{bmatrix} .30 \\ .20 \\ .50 \end{bmatrix}$$

But for each of these vectors, the numbers in the first row represent the number of subscribers to A, etc., so the vector of the total number of subscribers to A, B and None is

$$150000\begin{bmatrix} .70 \\ .20 \\ .10 \end{bmatrix} + 100000\begin{bmatrix} .15 \\ .80 \\ .05 \end{bmatrix} + 50000\begin{bmatrix} .30 \\ .20 \\ .50 \end{bmatrix}.$$

But this is exactly the matrix product PX written as a linear combination of the columns (see Theorem 5, page 55).

2. How does the matrix P^2 come into this?

Question: Are we computing $P(PX)$ or P^2X?

Answer: If we reason as before, the subscription vector after one year is PX. But we can take PX as a new starting point and ask what happens after one more year. Then that is the product of P with this vector, namely $P(PX)$. But by the associative law, the matrix product $P(PX) = (PP)X$, so we can find out what happens to vector X after two years by multiplying by the matrix product PP. This product is denoted by P^2. In this case,

$$P^2 = PP = \begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix}\begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix} = \begin{bmatrix} .55 & .24 & .39 \\ .32 & .68 & .32 \\ .13 & .08 & .29 \end{bmatrix}$$

Now one can check that if we multiply this matrix P^2 times X we get the same vector after two years, the same as $P(PX)$. So this is another way to get the answer to this problem: compute P^2 and then multiply this times X.

3. Noticing: Conservation and Markov Matrices

One thing to notice: The sum of the entries of X is 300,000. What is the sum of the 3 entries of PX, P^2X? Notice anything the same?

A second thing to notice: What is the sum of the 3 entries of each of the 3 columns of P? And of P^2? Notice anything the same?

Why does this happen? Well, since entries of the the first column of P add up to one, then if we multiply this column by 150,000, then the sum of the entries of the new vector will be 150,000. Likewise the sum of the entries of the two other subscription vectors will be 100,000 and 50,000. Since the sum of the entries of PX is the sum of all these 9 entries, the total will be 300,000, the same as the initial sum. This is true because the sum of the entries in the columns of P is equal to 1, for each column. Such a square matrix is called a *Markov Matrix*.

A slick way of seeing this using matrices, is to multiply by the matrix $E = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. For any column vector Y,

$$EY = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1 + y_2 + y_3$$

So this says that for X above, $EX = 300000$ and $EP = E$, the latter meaning that the sum of each column of P equals 1, so this is a matrix product way of expressing that P is a Markov matrix. But then using associativity, we can see that $EP^2E(PP) = (EP)P = EP = E$, so we see that P^2 is also a Markov matrix, as we already noted. But this means that the square of a Markov matrix is a Markov matrix. In fact, you can check the same way that the product of two Markov matrices is a Markov matrix.

Also, we see that $E(PX) = (EP)X = EX = 300000$, so the sum of the entries of the new subscription vector is the same as the original sum. This in effect means that our model neither creates or loses subscribers (no in-migration, no newly minted adults, no deaths). This is a feature (or limitation of this model.