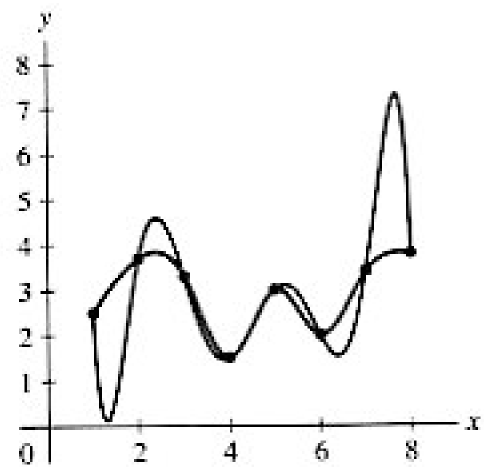
Fitting a Cubic Spline

If we have a large number of points that we want to fit a curve to, choosing a single polynomial is not wise. For such a large number of points, we would need a polynomial of high degree. But a polynomial of high degree usually has a great many maximums and minimums and therefore may have a very bumpy graph. A better choice of curve may be a *cubic spline*, in which we connect consecutive points with cubic (i.e., degree 3) polynomials. For example, here is the graph of the polynomial of degree 7 (thin black curve) that goes through a given set of eight points, together with the cubic spline (thick gray curve made up of seven adjacent cubics) that goes through the same eight points:



Observe that the graph of the polynomial swings well above and below the given points. On the other hand, the cubic spline not only avoids such wild swings but also its adjacent cubics fit together so smoothly that you cannot see where one cubic ends and the next begins. The smoothness of the spline is a consequence of the following "compatibility" condition, which is required of all splines:

> If the cubics $p(x)$ and $q(x)$ both go through a point (a, b), the first derivatives of p and q must be equal at $x = a$, and their second derivatives must also be equal at $x = a$.

This condition insures that adjacent cubics have the same tangent lines and same concavity at the points where they meet. Here is a simple example:

Example C Fit a cubic spline to the three points $P = (1, 3)$, $Q = (2, 7)$, and $R = (3, 4)$.

Solution First we clear the values of the variables a, b, and c, so we can use these letters again as variables:

- Clear the values of a, b, and c.

Then we introduce two cubic polynomials, p and q, with p defined on the interval $[1, 2]$ and q on $[2, 3]$:

- Define the function $p(x) = ax^3 + bx^2 + cx + d$.
- Define the function $q(x) = ex^3 + fx^2 + gx + h$.

Notice that there are altogether eight unknown coefficients in the two polynomials. Our goal is therefore to find a system of eight linear equations in terms of these unknowns. Solving this system will then give us the coefficients of the two cubics making up the cubic spline.

The polynomial p has to go through the points P and Q. This gives us the first two equations:

- Let *eqn1* be $p(1) = 3$:
- Let *eqn2* be $p(2) = 7$:

$$a + b + c + d = 3$$
$$8a + 4b + 2c + d = 7$$

The second polynomial q has to go through the points Q and R:

- Let *eqn3* be $q(2) = 7$:
- Let *eqn4* be $q(3) = 4$:

$$8e + 4f + 2g + h = 7$$
$$27e + 9f + 3g + h = 4$$

The remaining four equations are based on the first and second derivatives of the polynomials. We calculate the first derivatives of p and q, naming the new functions Dp and Dq, respectively:

- Let Dp = derivative of the function p:
- Let Dq = derivative of the function q:

$$Dp(x) = 3ax^2 + 2bx + c$$
$$Dq(x) = 3ex^2 + 2fx + g$$

In a similar way, we define the second derivatives, naming them DDp and DDq:

- Let DDp = derivative of the function Dp:
- Let DDq =derivative of the function Dq:

$$DDp(x) = 6ax + 2b$$
$$DDq(x) = 6ex + 2f$$

Since the graphs of both p and q go through the point $Q = (2, 7)$, they must satisfy the above compatibility condition; that is, the first derivatives of p and q must be equal at $x = 2$, and their second derivatives must be equal there as well:

- Let *eqn5* be $Dp(2) = Dq(2)$:
- Let *eqn6* be $DDp(2) = DDq(2)$:

$$\begin{aligned} 12a + 4b + c &= 12e + 4f + g \\ 12a + 2b \quad &= 12e + 2f \end{aligned}$$

So far we have only six equations for our eight unknown coefficients; we therefore need two more. One convention that is often used to provide the two missing equations is to require that the second derivatives of the spline equal zero at the leftmost and rightmost points. Thus,

- Let *eqn7* be $DDp(1) = 0$:
- Let *eqn8* be $DDq(3) = 0$:

$$\begin{aligned} 6a + 2b &= 0 \\ 18e + 2f &= 0 \end{aligned}$$

Now that we have eight equations, we solve the system in the usual way. However, we introduce a command that saves us from having to type in the 8 by 9 matrix:

- Let M = augmented matrix for the linear system consisting of the equations *eqn1* through *eqn8* and whose unknowns are the variables a, b, c, d, e, f, g, h:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 3 \\ 8 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 8 & 4 & 2 & 1 & 7 \\ 0 & 0 & 0 & 0 & 27 & 9 & 3 & 1 & 4 \\ 12 & 4 & 1 & 0 & -12 & -4 & -1 & 0 & 0 \\ 12 & 2 & 0 & 0 & -12 & -2 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 18 & 2 & 0 & 0 & 0 \end{bmatrix}$$

- Let s = solution of the linear system whose augmented matrix is M:

$$s = \begin{bmatrix} -7/4 \\ 21/4 \\ 1/2 \\ -1 \\ 7/4 \\ -63/4 \\ 85/2 \\ -29 \end{bmatrix}$$

Matching this solution with the list of unknowns [a, b, c, d, e, f, g, h] gives us the values for each of the coefficients. We then graph the cubic spline:

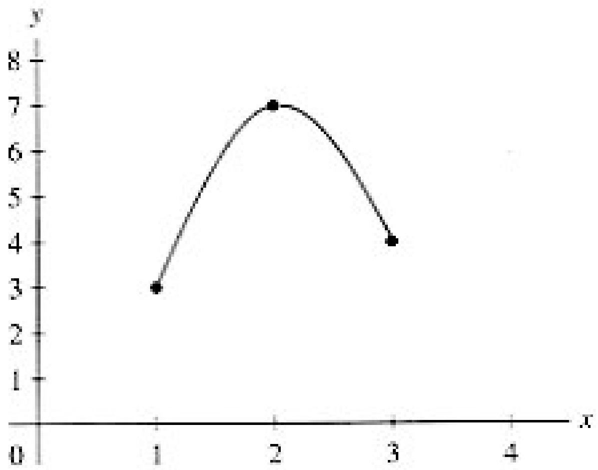
▸ Let *cubic1* = result of substituting s_1, s_2, s_3, s_4 for a, b, c, d, respectively, in $p(x)$:

$$cubic1 = -\frac{7x^3}{4} + \frac{21x^2}{4} + \frac{x}{2} - 1$$

▸ Let *cubic2* = result of substituting s_5, s_6, s_7, s_8 for e, f, g, h, respectively, in $q(x)$:

$$cubic2 = \frac{7x^3}{4} - \frac{63x^4}{4} + \frac{85x}{2} - 29$$

▸ Plot the points P, Q, R and the graphs of *cubic1* and *cubic2*:



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Explore The advantage of using cubic splines is only evident when we fit a large number of points. To explore visually what happens when we fit a spline to a large number of points, use the command below. The input is a list of points in the xy-plane. The output is a plot of the cubic spline (thick gray curve) through these points. You can include an optional argument that causes the interpolating polynomial (thin black curve) to be added to the picture. Execute the command below to see an example of a spline through five points and the interpolating polynomial through the same points. Change a point and observe how the curves change.

▸ Display, in a single plot, the points $(1, 3)$, $(2, 7)$, $(3, 4)$, $(5, 2)$, $(7, 1)$ and the cubic spline through these points and the interpolating polynomial through these points: