Sec 3.3, Problem 20

Let A be the matrix with columns v, w, so $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$. We are asked whether a certain vector b is in Sp(S), where S is the set {v,w}. In other words, is b in the range of A? This is the same as asking whether the equation Ax = b can be solved. This equation is the same as $x_1v + x_2w = b$.

So we can either solve each problem separately, or we can get a formula by letting $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Then reducing the augmented matrix yields this:

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 1 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 1 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}$$

Thus a vector b is in the span if $b_3 + b_2 - 2b_1 = 0$. If this equation is satisfied, then $x_1v + x_2w = b$ where $x_1 = b_1$ and $x_2 = b_1 - b_2$.

So the answers are

(a) 1 + 1 – 2 = 0, so $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1v + (2-1)w = v + w$

(b) -1 + 1 – 2 is not 0, so not in span.

(c) 0 + 2 - 2 = 0. We can use the formula, or we can just notice that this vector = v (thus = 1v + 0w).

(d) 1+3 – 4 = 0, so $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2v + (4-3)w = 2v + w$

Section 3.4, Problem 6

The coefficient matrix of this homogeneous system is A = $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. To find a basis of W, solve the equation Ax = 0. The solutions are all multiples of $\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ so this vector is a basis for W. (Thus W is a line through 0, a 1-dimentional subspace.)

Section 3.4, Problem 8

The coefficient matrix of this homogeneous system is A = $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. The solution set of Ax = 0 is the set of X of the form

$$\begin{bmatrix} -x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$. Thus a basis of W is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

S does not span because a set of two vectors cannot span all of 3-space. A vector b is in the span of S if we can solve the equation Ax = b, where the columns of A are the vectors in S. In other words, we are asking if b is in the range of A.

Section 3.4, Problem 12

(a) The matrix A reduces thus: $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) Null space is the set of all X = $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ so $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis of N(A).

(c) Since $A \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 0$.

If the columns of A are A_1, A_2, A_3, this says (-1) A_1 + (-1) A_2 + A_3 = 0 or

$A_3 = A_1 + A_2$. So we get the same set of all linear combinations (span) if we remove A_3 from the set. But { A_1, A_2}, so we stop here. This set is a basis of the column space.

(d) Basis for the row space is {[1 0 1], [0 1 1]}.

For details, see examples in this section and notes from Wed 10/25.

Section 3.4, Problem 14

(a) This matrix reduces to the 3x3 identity matrix I.
(b) The null space N(A) = the zero subspace = {0}. One can either say the zero subspace has no basis (see p. 195 after Def 4) or adopt the slick convention that a basis is the empty set {}. Either one is OK in this course, for now at least.
(c) A basis for the column space (range) consists of the set of all 3 columns of A.
(d) A basis for the row space could either be the rows of A or else the rows of I. (the standard basis of R^3).

Supplementary Problems, page 266, Problem 3

This equation Ax = 3x, if written out is a linear system in x1, x2, x3 with variables on both sides of the equation. Move all the variables to the left side and get a new matrix. This matrix is actually A – 3I. Call this matrix B

(a) This is N(B), thus a subspace.
(b) Find a basis of N(B) as in problems above and in examples

The reduce $B = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Vectors in null space of B are solutions of Bx = 0. This is set of these linear combinations: $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Thus a basis of N(B) = $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Since the basis has two elements, the dimension = 2.

Common sense note: Once we set up the matrix B, we see that the 3 equations are multiples of one equation in the 3 variables. Thus the solution is a plane, which has dimension 2.