Solutions to Problems 5.1 and 5.2 from Assignment 5

Problem 5.1

Let W be the set of vectors x in R^4 that are solutions to the equation x1 + x2 + x3 + x4 = 0. Let z = $[1, 2, 3, 4]^T$. Find vectors u and v so that z = u + v, where u is in W and v is orthogonal to all vectors in W.

Answer: Let N = $[1, 1, 1, 1]^T$. If X is any vector = $[x1, x2, x3, x4]^T$, then the equation above equates the dot product of N and X to zero. Thus W is the set of vectors orthogonal to N.

Also, W is the null space of the matrix N^T = [1 1 1 1] (see the definition of null space). Thus it is a 3 – dimensional subspace of R^4 since the rank of this matrix is 1, and so nullity = 3 = 4 – 1.

The question asks for u and v so that u is in W and v is orthogonal to the vectors of W.

If we compute orthogonal components of z, with v in the N direction, then u = z – v will be orthogonal to N, and thus in W.

(I will write the dot product of two vectors N and v as N.V, or as N^T v since I don't have a dot product symbol handy.)

By the formula in 3.6,
v = {(N^T v)/(N^T N)}N = {10/4)N = (5/2)N = $[5/2, 5/2, 5/2, 5/2]^T$.

So u = z – v = $[-3/2, -1/2, 1/2, 3/2]^T$

You can check that N^T u = 0.

Problem 5.2

Let S be the span of the vectors s1 = $[1, 0, 1, 1]^T$ and s2 = $[0, 1, 1, 1]^T$. Let w = $[1, 0, 0, 0]^T$. Find a vector t so that t is in S and w – t is orthogonal to both s1 and s2.

It is easy to check that {s1, s2} is an independent set, so it is a basis of S. Thus S is a 2-dimensional subspace.

To apply the projection formula of 3.6, we need an orthogonal basis, so we use the Gram-Schmidt process to convert s1, s2 to an orthogonal basis. This means that we write

s2 = u1 + u2, where u1 is in the direction of s1 and u2 is orthogonal to s1. Then {s1, u2} is a new basis for S, and it is orthogonal.

So by the formula, $u1 = \{(s1^T s2)/(s1^T s1)\}s1 = (2/3) [1, 0, 1, 1]^T$.
So $u2 = s2 – u1 = [-2/3, 1, 1/3, 1/3]^T = (1/3) [-2, 3, 1, 1]^T$.

Check that s1 and u2 are orthogonal and thus an orthogonal basis for S.
Simplifcation: We can multiply u2 by 3 and still get an orthogonal basis.
So let $s2^* = [-2, 3, 1, 1]^T$.
Then {s1, s2*} is also an orthogonal basis and is the one we will use.
Note: {s1, u2} works also, but the computation is marginally more complicated.

Now t is obtained from w by the projection (or coordinate) formula:

$$t = (s1^T w)/(s1^T s1)s1 + (s2^{*T} w)/(s2^{*T} s2^*)s2^* =$$
$$= (1/3) [1, 0, 1, 1]^T + (-2/15) [-2, 3, 1, 1]^T = (1/5) [3, -2, 1, 1]^T$$

One can check that this t is in fact (3/5)s1 + (-2/5)s2, so it is in fact in S.

Comment 1. Any orthogonal basis of S would work. If one realizes that since the lengths of s1 and s2 are equal, then the quadrilateral 0, s1, (s1+s2), s2 is a rhombus and the vector v1 = s1 + s2 is orthogonal to v2 = s1 – s2 (take the dot product and check), this gives a simpler computation.

Comment 2. We will learn a different method for solving this problem on Monday 11/13 when we explore section 3.8.