Solutions to Problems 5.1 and 5.2 from Assignment 5

Problem 5.1
Let W be the set of vectors x in $\mathbb{R}^4$ that are solutions to the equation $x_1 + x_2 + x_3 + x_4 = 0$. Let $z = [1, 2, 3, 4]^T$. Find vectors $u$ and $v$ so that $z = u + v$, where $u$ is in W and $v$ is orthogonal to all vectors in W.

Answer: Let $N = [1, 1, 1, 1]^T$. If $X$ is any vector $= [x_1, x_2, x_3, x_4]^T$, then the equation above equates the dot product of $N$ and $X$ to zero. Thus $W$ is the set of vectors orthogonal to $N$.

Also, $W$ is the null space of the matrix $N^T = [1 1 1 1]$ (see the definition of null space). Thus it is a $3$–dimensional subspace of $\mathbb{R}^4$ since the rank of this matrix is $1$, and so nullity $= 3 = 4 - 1$.

The question asks for $u$ and $v$ so that $u$ is in $W$ and $v$ is orthogonal to the vectors of $W$.

If we compute orthogonal components of $z$, with $v$ in the $N$ direction, then $u = z - v$ will be orthogonal to $N$, and thus in $W$.

(I will write the dot product of two vectors $N$ and $v$ as $N \cdot v$, or as $N^T v$ since I don't have a dot product symbol handy.)

By the formula in 3.6,
$v = \{(N^T v)/(N^T N)\}N = \{10/4\}N = (5/2)N = [5/2, 5/2, 5/2, 5/2]^T$.

So $u = z - v = [-3/2, -1/2, 1/2, 3/2]^T$

You can check that $N^T u = 0$.

Problem 5.2
Let $S$ be the span of the vectors $s_1 = [1, 0, 1, 1]^T$ and $s_2 = [0, 1, 1, 1]^T$. Let $w = [1, 0, 0, 0]^T$. Find a vector $t$ so that $t$ is in $S$ and $w - t$ is orthogonal to both $s_1$ and $s_2$.

It is easy to check that $\{s_1, s_2\}$ is an independent set, so it is a basis of $S$. Thus $S$ is a $2$-dimensional subspace.

To apply the projection formula of 3.6, we need an orthogonal basis, so we use the Gram-Schmidt process to convert $s_1$, $s_2$ to an orthogonal basis. This means that we write

$s_2 = u_1 + u_2$, where $u_1$ is in the direction of $s_1$ and $u_2$ is orthogonal to $s_1$. Then $\{s_1, u_2\}$ is a new basis for $S$, and it is orthogonal.
So by the formula, \( u_1 = \left( (s_1^T s_2) / (s_1^T s_1) \right) s_1 = (2/3) \left[ 1, 0, 1, 1 \right]^T \).
So \( u_2 = s_2 - u_1 = [-2/3, 1, 1/3, 1/3]^T = (1/3) \left[ -2, 3, 1, 1 \right]^T \).

Check that \( s_1 \) and \( u_2 \) are orthogonal and thus an orthogonal basis for \( S \).
Simplification: We can multiply \( u_2 \) by 3 and still get an orthogonal basis. So let \( s_2^* = [-2, 3, 1, 1]^T \). Then \( \{s_1, s_2^*\} \) is also an orthogonal basis and is the one we will use.
Note: \( \{s_1, u_2\} \) works also, but the computation is marginally more complicated.

Now \( t \) is obtained from \( w \) by the projection (or coordinate) formula:

\[
t = (s_1^T w)/(s_1^T s_1)s_1 + (s_2^*^T w)/(s_2^*^T s_2^*)s_2^* =
= (1/3) \left[ 1, 0, 1, 1 \right]^T + (-2/15) \left[ -2, 3, 1, 1 \right]^T = (1/5) \left[ 3, -2, 1, 1 \right]^T
\]

One can check that this \( t \) is in fact \( (3/5)s_1 + (-2/5)s_2 \), so it is in fact in \( S \).

**Comment 1.** Any orthogonal basis of \( S \) would work. If one realizes that since the lengths of \( s_1 \) and \( s_2 \) are equal, then the quadrilateral \( 0, s_1, (s_1+s_2), s_2 \) is a rhombus and the vector \( v_1 = s_1 + s_2 \) is orthogonal to \( v_2 = s_1 - s_2 \) (take the dot product and check), this gives a simpler computation.

**Comment 2.** We will learn a different method for solving this problem on Monday 11/13 when we explore section 3.8.