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Math 308
Project

Projections

Projections hold an important role in engineering, more specifically the software programs in which they utilize. Through mathematical calculations these programs project images onto a viewing plane, a window where you the viewer can see into a 3D environment. These programs are powerful tools where the user can create views within their viewing screen to project their design into a 2D image utilized for dimensioning or clarity of the part.

Boeing, Honda, Airbus among many other companies utilizes, now more than ever, computers to help design their products. They use various programs (AutoCAD, CATIA, ProE), with three purposes, to design a product more accurately, more expeditiously, and more economically sound. An engineer may create a solid model, a replica of what is to be built, in the 3D environment where complex surfaces and volumes determine the shape. These ***Computer-Aided Design*** programs are not only used to develop parts within the product but to help develop the product as a whole. In 1989, The Boeing Company completely defined its first jet in 3D-space within the CATIA environment (the 777). The purpose of such a tool, which has been denoted as EPIC (Engineering Pre-Assembly In CATIA) is not only to see how the aircraft will look, but to check for interferences from part to part, as well as to coordinate from engineer to engineer how their parts fit together and how those parts function. By utilizing CATIA Boeing managed to save an engrossing amount of resources and materials. Boeing is just one of numerous companies who utilize such tools. Designing products, whether it is aircrafts, boats, cars, or even the tools used to build such products, tends to be the standard of today.

These CAD programs not only allow the user to create three dimensional objects but grant the user the capability to represent those solid or wireframe images into a two dimensional image, a projection onto a plane. This is necessary for the engineering companies to draw there blueprints as to how the part is to be built, dimensionally as well as structurally.

The key to understanding how CAD programs transform 3D elements into 2D elements is through projections. An object in the 3-dimensional space can be projected point by point onto the surface of a plane. If a line is drawn from every point of the object to a common focus point "spotlight" and the intersections are linked together in the plane a projection of the object onto the plane is made. The position of the viewer determines the nature of the projection. If the viewer is infinitely far away the lines starting from different points of the object are parallel, this is parallel projection. If the viewer is not infinitely far away, the beams coming from different points are coming from different directions, every line makes a different projection (projection onto the plane along the line), this is perspective projection. The plane in 3D space can be defined as a subspace if the system of coordinates is selected so that the origin belongs to the plane. For example, if there were to be an object, such as a prism, in (x,y,z) Cartegian coordinates (sometimes referred to as space coordinates) then it would be possible, by altering its matrix, to create a two dimensional view. In fig. 1, a prism with coordinates (2,0,0), (1,0,1), (1,2,1), (1,1,0) is being projected onto the yz plane. Imagine a spotlight at some arbitrary position beyond that of the object along the x-axis, then the projection would be the "shadow" of the object. Each point, which is to be mapped onto the plane, follows a straight path from the spotlight, or ***center of focus,*** to the plane. If the spotlight were not to be on the x or y or z-axis, the projection could be created by first translating the coordinates until the center of focus is where the origin is.

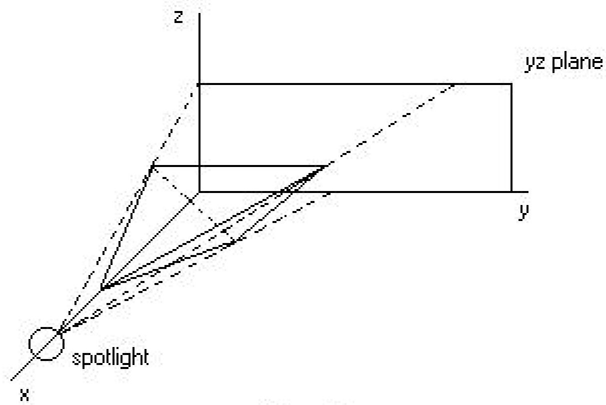


Figure 1

Figure 2 demonstrates how it is possible to determine a point on the plane by using similar triangles. By placing the center of focus at (3,0,0), then the point (1,1,0) projects onto the plane at point (0,1.5,0).

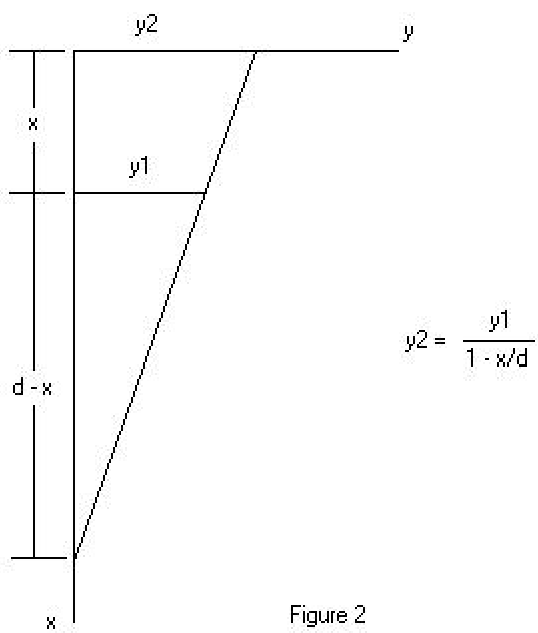


Figure 2

It turns out that the other points of the prism hold to the same principle.

$$x2 = \frac{x1}{1 - x/d}$$

$$z2 = \frac{z1}{1 - x/d}$$

Point (2,0,0) becomes (0,0,0), point (1,0,1) becomes (0,0,1.5) and point (1,2,1) becomes (0,3,1.5). The equations for y2, x2, and z2 can be converted into a matrix P. For that to happen it is necessary to transform the space coordinate points into homogeneous points, such as, (2,0,0,1), (1,0,1,1), (1,2,1,1), and (1,1,0,1). The matrix P that will transform these points onto the plane is

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 1 \end{vmatrix}$$

Since the projection plane in figures 1 and 2 are of the yz plane, x is 0 in the matrix. This matrix only holds for special cases where the focus point is located along an axis. If the center of focus were to be located at a position where none of the x, y, and z coordinates have a zero value then another matrix will need to be implemented; the geometry of similar triangles may not apply.

Let there be an object, such as in figure 3, where M are the vertices with a center of focus (0,0,8).

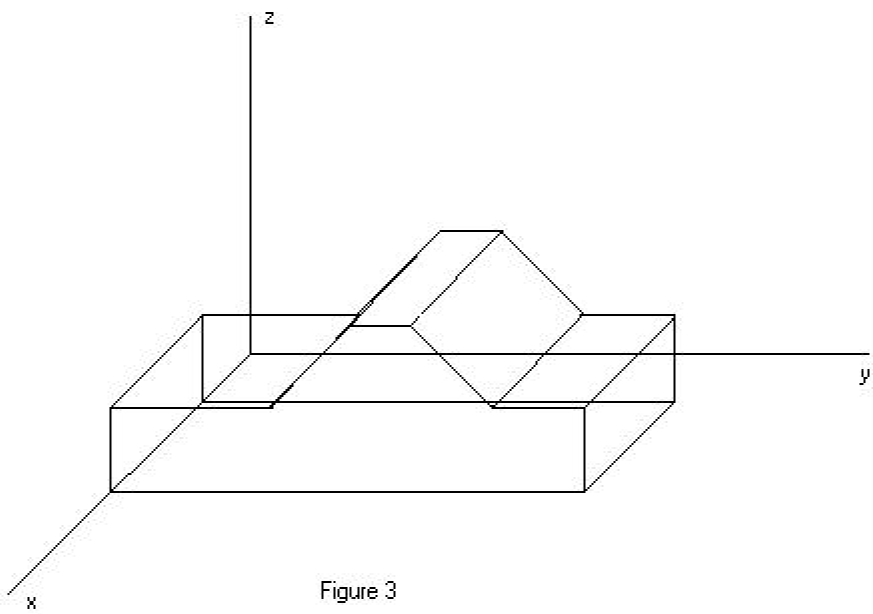


Figure 3

$$M = \begin{vmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 4 & 5 & 6 & 6 & 0 & 0 & 2 & 3 & 4 & 5 & 6 & 6 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

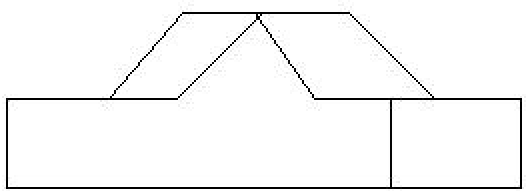
By computing PM, the result is

$$PM = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 4 & 5 & 6 & 6 & 0 & 0 & 2 & 3 & 4 & 5 & 6 & 6 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ .625 & .625 & .625 & .625 & .625 & .625 & .625 & .625 & .875 & .875 & .875 & .875 & .875 & .875 & .875 & .875 \end{vmatrix}$$

PM will need to be converted into 3D coordinates to allow the new generated points to be projected into the plane. For this to happen the x, y, z points will need to be divided by the fourth row. For example, the second row, third column (third point, y-coordinate) has the value of 2, which is to be divided by .625, the new value, and actual y-coordinate is 3.2. In essence, the coordinates must transform back into homogeneous coordinates for the points to be plotted. By doing so the vertices of transposed object is

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.2 & 4.8 & 6.4 & 8 & 9.6 & 9.6 & 0 & 0 & 2.29 & 3.43 & 4.57 & 5.71 & 6.86 & 6.86 \\ 0 & 1.6 & 1.6 & 3.2 & 3.2 & 1.6 & 1.6 & 0 & 0 & 1.6 & 1.6 & 3.2 & 3.2 & 1.6 & 1.6 & 0 \end{vmatrix}$$

which on the yz plane has the appearance of



CAD systems utilize various matrixes depending upon the location of the focus and the location as well as orientation of the plane to determine the projection. Whatever matrix is chosen, linear algebra is the key to the transformation. These engineering tools use parallel as well as perspective projection to display the wireframe or solid model. If the part is to be dimensioned with orthogonal, or orthographic views than it is necessary for the program to use parallel, while if the part is to be a 3D image than it is necessary for it to be done by perspective. It is interesting for me, since I work on a CATIA machine daily, to see how programs manipulate numbers and matrixes so objects are displayed on the screen. Obviously there is more in the process than linear algebra to show what we see on the screen, but it is a key step.

Bibliography:

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