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Using Eigenvalues to Find Solutions to Dampened Spring Problems

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Abstract: By reducing the differential equation for simple harmonic motion into two simplified differential equations and then using the resultant equations to form vectors, a matrix can be developed whose eigenvalues reduces to a simple linear equation. This equation is then the solution to simple harmonic motion and can be transformed into an exponential equation of two components. The first component represents the sinusoidal nature of harmonic motion while the second component represents a value for damping the oscillations to form a diminishing sin wave.

Explanation:

m = mass attached to spring
x = height (from 0) of point on sin wave
b = damping constant
k = restoring force on spring

The second order differential equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

is the equation for simple harmonic motion where x double-dot is acceleration and x dot is velocity. If we then set

$$\dot{x}_1 = x$$

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1$$

we can insert these values into a matrix to form:

$$\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix} = \begin{matrix} 0 & 1 \\ -k/m & -b/m \end{matrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$$

Then, we can set up the matrix to find the eigenvalues:

$$\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix} = \begin{matrix} (0-\lambda) & 1 \\ -k/m & \left(-b/m - \lambda\right) \end{matrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$$

Taking the determinant, we find that:

$$(\lambda - 0)\left(\lambda + b/m\right) + \left(k/m\right) = 0$$

Multiplying through gives:

$$\lambda^2 + \left(b/m\right)\lambda + \left(k/m\right) = 0$$

Using the quadratic formula and simplifying results in:

$$\left(-\frac{b}{2m}\right) \pm \sqrt{\left(\frac{b^2}{4m^2}\right) - \left(\frac{k}{m}\right)}$$

Since the solution to any system of linear differential equations is an exponential equation, the simplified quadratic formula yields:

$$Ae^{\left(-\frac{b}{2m}\right) \pm i\sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}\,t}$$

Which, through the properties of exponential equations, can also be expressed as:

$$Ae^{\left(-\frac{b}{2m}\right)}e^{\pm i\sqrt{\frac{b^2}{4m^2} - \left(\frac{k}{m}\right)}\,t}$$

Where the first exponent is the equation for simple harmonic motion, the second exponent is the equation for the damping coefficient and *A* is the initial displacement. If we denote the angular frequency to be ω′ (as opposed to ω, the angular frequency for simple harmonic motion) and set it as:

$$\omega' = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

This allows us (through some mathematical intuition) to greatly simplify the expression to:

$$x = Ae^{-\left(\frac{b}{2m}\right)} \cos \omega' t$$

Giving us an easily used exponential equation for the displacement of a mass attached to a damped spring with respect to time.

Example: A car has a mass of **1360 kg**. When the front wheel hits a **3 cm** bump, the suspension should compress the full 3 cm but then the we want the spring action to be damped so that the oscillations, which occur at a rate of **1.5 rad/s**, diminish to zero in 2 seconds. What damping value should be used?

First, we set x to zero so that the oscillations diminish to that point. This gives us:

1.

$$0 = 3_{cm}e^{-\frac{b}{2(1360kg)}2s} \cos(1.5rad)2s$$

2.

$$-.141 = 3_{cm}e^{\frac{b}{2720kg}2s}$$

3.

$$.047 = e^{-\frac{b}{2720kg}2s}$$

4.

$$\ln .047 = \ln e^{-\frac{b}{2720kg}2s}$$

5.

$$-1.52 = {}^{-b}\!\diagup_{2720kg}$$

6.

$$b = 4158.34$$

Conclusion: The difficulty of solving this problem and countless others like it is greatly reduced when the exponential equation method is used in contrast to the differential equation that forms the basis of the method just described. Without the eigenvalue reduction method outlined above, the problem would take on an unneeded complexity that could easily result in a miscalculation and/or loss of time in computations.

Bibliography

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