MATH 308A PROJECT:

An Interesting Application of Linear Algebra

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Math 308 A
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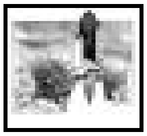
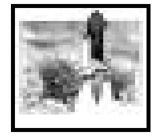


Application: Space Shuttle Control Systems



Abstract:

This report discusses the ways in which Linear Algebra is applied to the manipulation of an object in three-space. This topic has a variety of useful applications, in fields ranging from Computer Animation to Aerospace Engineering. Specifically, the "object" considered in this paper is the Space Shuttle. The linear algebra topics necessary for this analysis include orthogonal coordinate systems and rotation matrices.

Introduction:

Throughout our Math 308 course, we have seen several possible areas of application for linear algebra in real life. David C. Lay discusses several such applications in his text, Linear Algebra and Its Applications. One of the most fascinating areas in which linear algebra is utilized involves the rotation of an object in three-space, a topic that is widely used in today's modern world. One specific object that requires the use of linear algebra is the Space Shuttle. I sought to find out the answer to the question: How is an object like the Space Shuttle manipulated in three-space? In today's modern world, the answer to this may seem to be old news. NASA scientists and engineers have obviously mastered the solution, but I would venture a guess that most of "the rest of us" have not thought much about this at all. In seeking to answer this question, I discovered this is not so simple a task as NASA makes it seem!

This document outlines the process of 3-D manipulation using simple examples. Clearly, real-life situations would involve much more complex cases than given here. In creating this report, the Maple V program was used for matrix calculations. Please refer to the Example Appendix in the back to view the work done by Maple.

Overview:

One of the challenges a Space Shuttle Control Pilot might face is how to get the Shuttle from one point to another. In other words, the pilot has some desired "place" in three-space for the Shuttle to ultimately arrive at. The goal here is to be able to impose some sequence of controls so that the Shuttle arrives at some target vector. Essentially, the pilot is trying to get the Shuttle's Control System to deliver this desired output vector. The reader should note that this report will *not* explain how to obtain the desired sequence of controls in order to achieve a given output vector for the Shuttle, since those details would become quite lengthy. Instead, this report focuses on the necessary concepts to understand *before* one can go about creating the "controllability matrix," namely, obtaining an orthogonal vector set for defining a coordinate system and the mathematics of the 3D rotation matrix. The combination of these two concepts and some additional linear algebra will ultimately yield in one's ability to derive a sequence of matrix operations that *will* allow one to manipulate the Shuttle and other objects in three-space to achieve a desired result.

Q: How can we manipulate an object in three-space?

A: There appear to be two main concepts to understand before solving this problem. First, we must **define a coordinate system** for our object. One requirement is that each vector in this system must be orthogonal, that is, perpendicular. Mathematically speaking, this means that all the combinations of pairs of our vectors must have a dot product of zero. This topic will be further explored in the next section. Secondly, we need to be able to **rotate our object about each of the axes** in our coordinate system. This involves the use of the rotation matrix as defined for the 3x3 case. In this Space Shuttle application, only the rotation matrix for the 3x3 case is considered.

Defining A Coordinate System

The derivation of three axes in a coordinate system is certainly not a new invention. The concepts have been used since the days of sailing, and are used more frequently now in the 21^{st} century, with the growing popularity of aviation.

It is important to understand that the coordinate system we define is "fixed." When we say that the system is "fixed," we mean not that it is fixed to the object (i.e., the Space Shuttle), but that its orientation is fixed in space. This idea will become clearer after considering rotations about each of the axes, as discussed in the next section.

As previously mentioned, a coordinate system that is useful for manipulation must be orthogonal. In terms of matrices, this is equivalent to saying that, "each pair of distinct vectors from the coordinate set is orthogonal" (Lay text, page 379).

For example, suppose we have a set of three vectors {**i**, **j**, **k**}. To find whether or not this set is orthogonal, we consider the three possible pairs of vectors, namely, {**i**, **j**}, {**i**, **k**}, and {**j**, **k**}. Is the dot product of each of these vectors zero? For this calculation, refer to <u>Example A</u> in the Example Appendix.

<u>Example A</u> shows that when the dot products of each pair of vectors in our set are zero, we do in fact have an orthogonal set. Once we define an orthogonal set for our object, we can use these vectors as our coordinate system.

Once we orient our object in an orthogonal coordinate system, we need to be able to manipulate and control it. One way to do this is to rotate our object about the each of the axes we just defined as our coordinate system. In terms the Space Shuttle, the ability to rotate translates into being able to point the Shuttle in the direction we want it to go. The next section describes such methods of rotation.

Rotations in Three-Space

The rotation matrix for the 3x3 case is defined as follows:

Rotation around the x-axis:
Counterclockwise rotation of gamma about the about the x-axis

$$R_X(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$

“ROLL”

Rotation around the y-axis:
Counterclockwise rotation of beta about the y-axis

$$R_Y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

“PITCH”

Rotation around the z-axis:
Counterclockwise rotation of alpha about the z-axis:

$$R_Z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

“YAW”

The terms “roll, pitch, and yaw” are used to describe the rotations about the x-, y-, and z- axes of the fixed coordinate system, respectively.

Each rotation matrix for the 3D case is a simple extension of the 2D rotation matrix. For example, the yaw matrix essentially performs a 2D rotation with respect to the x-y plane, while leaving the z- coordinate unchanged. The third row and third column look like part of the identity matrix, while the upper left portion of the yaw matrix looks like the 2D rotation matrix

Rotation matrices can be multiplied with each other, which leads to a series of rotations of an object around the x-, y-, and z- axes. There is one crucial fact to note when dealing with rotation matrices: ***The order of rotations is important****, which means it is not commutative.*

For example, applying a rotation of angle α around the x-axis, followed by a rotation of angle θ around the z-axis results in the given rotation matrix for the final orientation of the body:

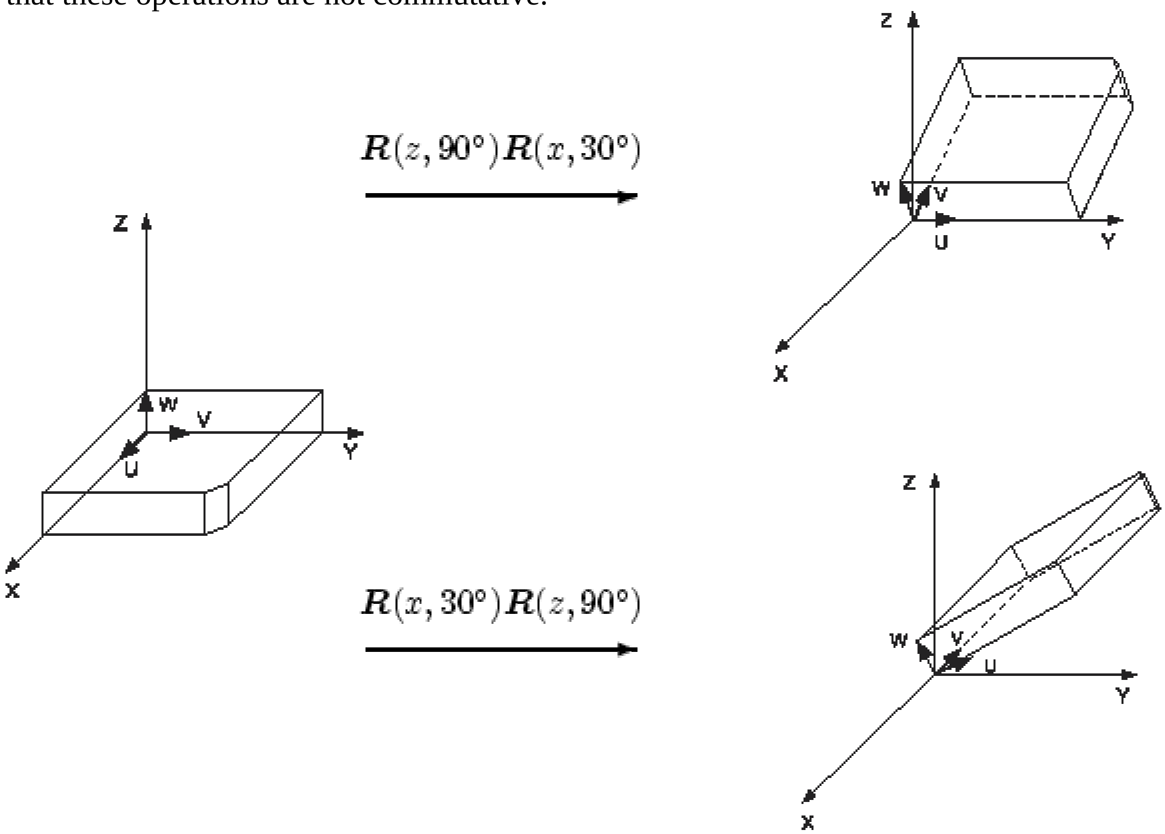
$$\boldsymbol{R} = \boldsymbol{R}(z,\theta)\boldsymbol{R}(x,\alpha) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

However, starting with a rotation of the same angle θ around the z-axis as before, followed by a rotation with the same α around the x-axis, the result is the following rotation matrix:

$$\boldsymbol{R} = \boldsymbol{R}(x,\alpha)\boldsymbol{R}(z,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These two combinations do not yield the same result. To further understand this, consider the following series of rotations:

In the picture below, the figure at left is first rotated 90 degrees about the z-axis, then rotated 30 degrees about the x-axis. The result is shown in the top right figure. Next, the same operations are performed *but in the opposite order*, as shown in the bottom right view. It is now obvious that these operations are not commutative.



Putting It All Together

Q: What does this mean for the Space Shuttle?

We have seen how to create orthogonal vector sets and how to manipulate vectors in three-space. Understanding these two basic concepts is the first step; one can now expand on them to derive a sequence of control operations to use for directing 3-dimensional objects such as the Space Shuttle. This is one of the greatest benefits to be had from the application of Linear Algebra. It is clear that these concepts have wide-ranging applications to real-world problems in today's modern society. Some of the most challenging marvels of engineering and science have been made possible by utilizing concepts from this type of mathematics.

BIBLIOGRAPHY:

The information in this report was obtained from the following sources:

Lay, David C. Linear Algebra and Its Applications. 2000. Addison Wesley Longman, Inc. pp.209, 210, 367-386.

http://prt.fernuni-hagen.de/lehre/KURSE/PRT001/course_main/node9.html

http://phoenix.nmt.edu/~bruder/ee448/ee543/vol2.pdf

http://www.itee.uq.edu.au/~elec3700/ELEC3700_M4.pdf

http://www.math.duke.edu/education/ccp/materials/linalg/rotation/rotm3.html

http://msl.cs.uiuc.edu/~lavalle/cs497/book/geom/node10.html