

Linear Programming and the Simplex Method

Abstract

This article is an introduction to Linear Programming and using Simplex method for solving LP problems in primal form.

What is Linear Programming?

Linear Programming is the method of finding an optimal solution for a linear function F of n variables, when the variables are under some *linear* constraints.

Here we introduce two famous problems in the field of linear programming:

Production Problem

The production problem involves determining the *rate of production* for n products, so that the profit resulting from the products is maximum. Each product uses some *resources*, which are finite and provide the constraints. Here is a formal statement of the production problem:

Products P_1, \dots, P_n are produced from resources R_1, \dots, R_m . b_i is the amount of resource R_i available to the manufacturer. a_{ij} is the number of units of R_i needed for one unit of P_j . The profit from producing one unit of P_j is known to be c_j .

So the problem is to maximize

$$f(x_1, \dots, x_n) = \sum_{j=1}^n c_j x_j \quad (x_j \text{ representing the amount of } P_j \text{ produced})$$

under the conditions

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for all } i, \quad 1 \leq i \leq m$$

(and every x_j non-negative).

Diet Problem

The diet problem is determining the least costly amount of food to be consumed, in order to satisfy a minimum daily requirement of some nutrients (given the amount of nutrients for each unit of food and the cost of each food unit).

Primal Form

In order to find a simple algorithm for solving LP problems, we need a standard form for the problem to work with. This is called the primal form and it can be shown that every LP problem can be transformed to a corresponding problem in the primal form.

Although we might want to find a minimum for the function F , problems in primal form concern finding a maximum for F (which is called the **objective function**). Minimizing $F(\lambda)$ is clearly equal to maximizing $F(-\lambda)$.

The constraints in LP problems in primal form are only in the form of
$$\sum_{j=1}^n a_{ij}x_j \leq b_i .$$

Constraints using $>$ or $=$ can be transformed to this form.

All variables are non-negative in the primal form. (Although this is usually the case in LP problems, when we need negative variables we can replace them with $x' - x''$, where x' and x'' are non-negative.

Feasible Solutions

Any (x_1, \dots, x_n) satisfying the *constraints* of the LP problem is said to be a feasible solution for that problem. If at this solution the objective function reaches its required maximum or minimum, the solution is said to be *optimal feasible solution* (OFS).

The Solution of an LP Problem

In order to solve an LP problem, we must first find out whether at least one feasible solution exists or not. If it exists, we must either find the OFS or show that it does not exist (this happens when the objective function is not bounded in the required direction).

Example of an LP problem

- Maximize $f(x_1, x_2) = -2x_1 + 5x_2$ if

$$x_1 + x_2 \leq 5,$$

$$2x_2 \leq 4,$$

First we see that (0,0) is a feasible solution. For all x_1, x_2 , $f(x_1, x_2) = -2x_1 + 5x_2 \leq 5x_2$.

10. (since x_1, x_2 are non-negative). So 10 is the upper bound of the objective function, and since $f(0,2) = 10$, (0,2) is the OFS.

Simplex Method

The Simplex algorithm for solving LP problems was discovered in 1947 by George B. Dantzig. Nearly all practical LP problems are solved by a variation of the Simplex method.

We introduce here some notions that are useful in implementing Simplex method.

Condensed Tableau

The condensed tableau is a representation of a system of linear equations which is in reduced-row echelon form. It is based on the contrast between pivot variables and non pivot variables.

The condensed tableau below represents the system of linear equations:

$$x_{j_1} + \sum_{q=1}^p a_{1q} x_{k_q} = \beta_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$x_{j_m} + \sum_{q=1}^p a_{mq} x_{k_q} = \beta_m$$

$$k_1 \quad k_2 \quad \dots \quad k_p$$

a_{11}	a_{12}	\dots	a_{1p}	b_1	j_1
a_{21}	a_{22}	\dots	a_{2p}	b_2	j_2
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
a_{m1}	a_{m2}	\dots	a_{mp}	b_m	

j_m

What we have done, basically, is to eliminate the coefficients for the pivots, (since they are 1). The numbers $j_1 \dots j_m$ are the indices for pivot variables and $k_1 \dots k_p$ the indices for free variable.

Now we explain how to change the role of a pivot and free variable. Suppose we want to change the place of variables k_q and j_i (the first a free variable and the second a pivot).

All the elements of the tableau should change in this manner:

	k_q		j_r		
	\vdots		\vdots		
.....	a_{rq}/a_{rs}	$1/a_{rs}$	b_r/a_{rs}	k_s
	\vdots		\vdots		
.....	$a_{iq} - a_{is} a_{rq}/a_{rs}$	a_{rq}/a_{rs}	$- a_{iq}/a_{rs}$	$b_1 - a_{is} b_r/a_{rs}$	j_1
	\vdots		\vdots		
				\vdots	

We can easily see that this is a normal row operation, and thus prove the method easily.

Simplex Tableau

Here we introduce some changes in the primal form of LP problems that makes it fir for using Simplex method.

First we introduce *slack variables*. Slack variables are introduced to change all the inequalities to equations. It is based on the observation that $a \leq b$ means there is some non-

negative c for which $a+c=b$. So each constraint $\sum_{j=1}^n a_{ij}x_j \leq b_i$ can be changed to

$\sum_{j=1}^n a_{ij}x_j + x_{n+i} = b_i$. Now the only restriction on the primal form to make it fit for Simplex

method is that all b_i are non-negative.

So we introduce slack variables for each equation (which is clearly pivot), and form the following form, which has a row added for the objective function and is called the Simplex tableau.(The indices remain the same, except for the objective row).

a_{11}	a_{12}	\dots	a_{1p}	b_1
a_{21}	a_{22}	\dots	a_{2p}	b_2
\vdots	\vdots	\vdots	\vdots	\vdots
a_{m1}	a_{m2}	\dots	a_{mp}	b_p
$-c_1$	$-c_2$	\dots	$-c_n$	0

Since all b_i are non-negative, we see that $(0, \dots, 0)$ is a feasible solution (and the item at the last column, last row, which is called the corner number gives the $F(0, \dots, 0)$). Now it can be shown that 1. The pivoting process explained above remains unchanged on the *objective row* 2. If we select appropriate entries in the tableau, by doing the pivoting operation, we get a corner number greater (or equal) to the current one. In other words, we get closer to the maximum.

The basic solution of a tableau is the solution obtained from assigning 0 to all non-basic variables(whose index is given at the top of the tableau).

The rule is to choose a negative entry in the objective row. For every positive a_{ij} above it, calculate a_{ij}/b_i .(this is called the θ -ratio). Choose the entry with the least θ -ratio (if there is a tie, choose one of them). Pivot that entry (The purpose of the θ -ratio is to keep the b_i for all i non-negative, so that we can use the method in the next iteration). We can prove that the corner number is greater or equal to the previous one. The only case when it is equal is when we have a $b_i=0$. This leads to a basic variable being 0 in the basic solution , which is called degenerate case.

Now we can explain the Simplex algorithm.

1. See whether a negative entry exists in the objective row. If not, the basic solution is the OFS (the corner number being the maximum).
2. If every negative entry in the objective row has at least one positive number in it's column, the objective function is unbounded above.
3. Choose a negative entry in the objective row, choose a positive entry in it's column which has the least θ -ratio. Pivot the tableau at this entry. Go to 1.

It is clear that the Simplex algorithm allows a degree of freedom in choosing the entry to be pivoted, and this leads to different methods for implementing it.

In some cases, we might be stuck with a Simplex algorithm that returns to the same tableau again and again (and thus, never stops). The following selection method guarantees that we don't have to face this problem.

- Always choose the element with the smallest subscript, when two or more choices exist.

We must use this both for choosing entries in the objective row and the positive integer above it. It's called Bland's Termination Theorem.

Simplex Method in Practice

1	2	
1	3	4
-1	2	0
1	4	5
-3	-2	0

1	2	
1	3	4
-1	2	0
1	4	5
-3	-2	0

3 2

1	3	4
1	5	4
-1	1	1
3	7	12

So the answer is (4,0).