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Math 308A

Ellipse and Linear Algebra

Abstract

Linear algebra can be used to represent conic sections, such as the ellipse. Before looking at the ellipse directly symmetric matrices and the quadratic form must first be considered. Then it can be shown, how to write the equation of an ellipse in terms of matrices. For an ellipse that is not centered on the standard coordinate system an example will show how to rotate the ellipse.

Symmetric Matrices

The symmetric matrix is a matrix in which the numbers on ether side of the diagonal, in corresponding positions are the same.

$$\begin{bmatrix} 1 & 5 \\ 5 & 8 \end{bmatrix} \qquad \begin{bmatrix} 9 & -2 \\ -2 & 7 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Examples:

Quadratic Form

Now we have seen the symmetric matrices, we can move on to the quadratic

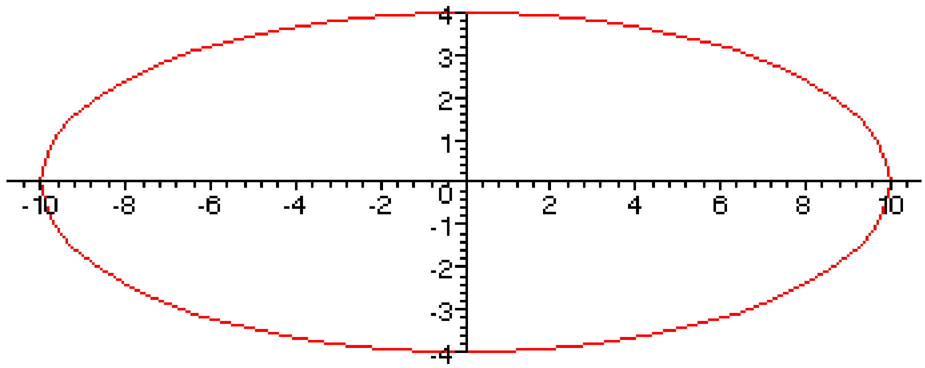
$$Q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2$$

form. The quadratic form is the function, $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where $\mathbf{x}$ is a variable vector.

Ellipse

An ellipse has a the standard equation form:

$$\frac{x^2}{a^2}+\frac{y^2}{c^2}=1$$



$$\frac{x^2}{10^2}+\frac{y^2}{4^2}=1$$

Change Variable

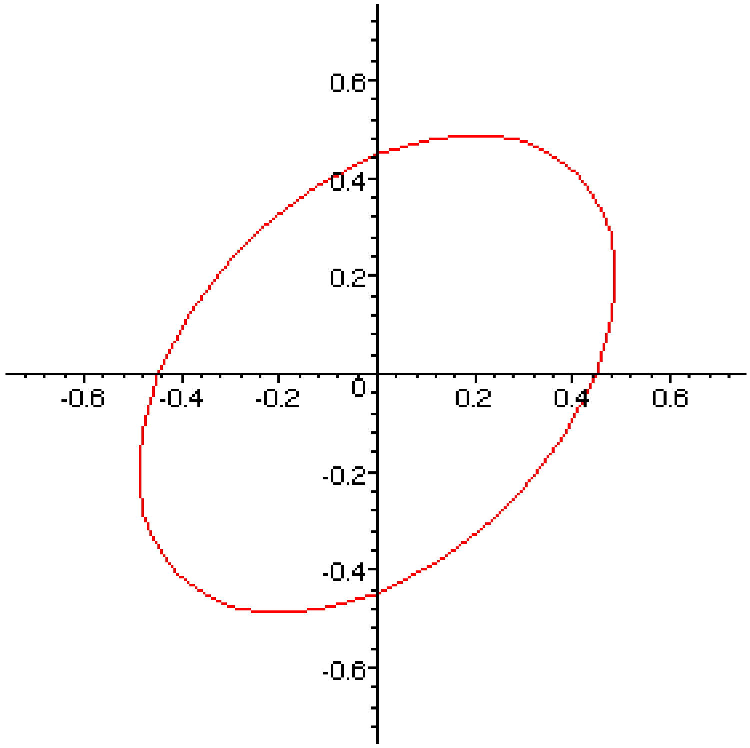
Before we can rotate an ellipse we first need to see how to change the variable vector. By changing the variable ellipses in non standard form can be changed into standard form. By using the function Q(**x**) (expanded) the x_1x_2 term, determines how much rotation an ellipse has relative to the standard coordinates, so by changing the variable in affect we are changing the coordinate system.

The Principal Axes Theorem:

Let Abe an n x n symmetric matrix. Then there is an orthogonal change of variable, $\mathbf{x}=P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic from $\mathbf{y}^T D\mathbf{y}$ with no cross-product term (x_1x_2) (Lay, 453).

Example: Ellipse Rotation

Use the Principal Axes Theorem to write the ellipse in the quadratic form with no x_1x_2 term.



$$5x_1^{\ 2} - 4x_1x_2 + 5x_2^{\ 2} = 1$$

Solution:

1. Write the matrix A for the equation:

$$5x_1^{\ 2} - 4x_1x_2 + 5x_2^{\ 2} = 1$$

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

2. Find the eigenvalues for A.

$$\lambda = 7,3$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \lambda = 7 \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 3$$

3. Use the eigenvalues to find the eigenvectors.

4. Check to make sure the two eigenvectors are orthogonal, if the dot product equals zero then the two vectors are orthogonal.

$$\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

5. A = PDP^{-1}:

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

6. $\mathbf{x} = P\mathbf{y}$,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

7. Compute

$$5x_1^{\,2} - 4x_1x_2 + 5x_2^{\,2} = \mathbf{x}^{\mathrm{T}}A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$

8.

$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 7y_1 \\ 3y_2 \end{bmatrix} = 7y_1^{\,2} + 3y_2^{\,2}$$

9. Therefore

$$5x_1^{\,2} - 4x_1x_2 + 5x_2^{\,2} = 7y_1^{\,2} + 3y_2^{\,2}$$

Bibliography

Banchoff, Thomas, and John Wermer. Linear Algebra Through Geometry, Springer-Verlag, New York Inc., New York, 1993.

Lay, David C., Linear Algebra and Its Applications, 2nd ed. updated, Addison Wesley Longman, Inc. New York, New York, 2000.