**ANSWERS to Assignment 4 (Due Monday 1/31)**

**Reading**: Gemignani, Chapter 6 and Section 7.1.

**Problem 4.1**

For each of these sets, list all the subsets of the set. Also, write down the number of elements of the set and the number of subsets.

a) Set A = {}.

A has zero elements and one subset, which is A itself (the empty set)

b) Set B = {1}.

B has 1 element and two subsets: {}, {1}.

c) Set C = {1, 2}.

C has 2 elements and 4 subsets: {}, {1}. {2}. {1, 2}.

d) Set D = {1, 2, {1, 2}}

D has 3 elements and 8 subsets:

{}, {1}. {2}. {{1, 2}}, {1, 2}, {2, {1, 2}} , {1, {1, 2}} , {1, 2, {1, 2}} .

**Problem 4.2:** Gemignani, Section 6.2 # 3

Denote by N the set of positive integers. Let T be the set of all unending decimals of the form

$k = .k_1k_2k_2 \ldots$

where each $k_i$ is either 0 or 1 (as in Example 4). Let P be the set of subsets of N.

Prove that P has the same number as the set T.

Proof. To prove this one needs to set up a one-to-one correspondence $\psi$ between the subsets of N and the real numbers in T.

Here is how a correspondence $\psi$ from P to T can be defined:

Any subset S of N is a set of positive integers. Define a sequence of integers $k_1, k_2, k_2 \ldots$ be defined to be $k_i = 1$ if $i \in S$ and $k_i = 0$ if $i \notin S$. Then let this sequence be the numbers of the decimal expansion of a number, which will be in T, since these integers are all either 0 or 1. This defines a number $\psi(S)$ in T for any $S \in P$.

For the correspondence $\omega$ in the other direction, start with a number k in T and let $k_1, k_2, k_2 \ldots$ be the sequence of 0s and 1s forming the decimal expansion of k, Then define a subset of N by this condition: an integer $j \in N$ is in S if and only if $k_j = 1$. This set is $\omega(k)$.

To check that this is really a one-to-one correspondence, we note that $\psi$ and $\omega$ are inverses of each other. Start with a set S in P. Apply $\psi$ and then $\omega$; from the definitions you get the same set S. Likewise it is true that if you start with a number r in T and apply $\omega$ and $\psi$, the number resulting is the original number r.

**Problem 4.3:** Gemignani, Section 6.2 # 4

Suppose S is a set with n elements and T is a set with m elements.

a) What is the smallest number of elements that $S \cup T$ can contain?

Answer: Since $S \subset S \cup T$ and also $T \subset S \cup T$, then $n \le \#(S \cup T)$ and $m \le \#(S \cup T)$, because the cardinal number of a subset is less than or equal to the cardinal number of the containing set. Thus $\max(m, n) \le \#(S \cup T)$.

In fact max(m, n) is the smallest possible value of $\#(S \cup T)$. To show this, it suffices to produce one example with this value, for a given choice of m and n. As the example, let $S = \{1, \dots, n\}$ and $T = \{1, \dots, m\}$. Then $S \cup T = S$ if $m \le n$ and $S \cup T = T$ if $n \le m$. In the first case, $\#(S \cup T) = n = \max(m, n)$ and in the second case $\#(S \cup T) = m = \max(m, n)$. So max(m, n) is the smallest number possible.

b) What is the largest number of elements that $S \cup T$ can contain?

Answer: If S and T are disjoint, then $S \cup T$ has m+n elements.

c) What is the largest number of elements that $S \cap T$ can contain?

Answer: Since $S \supset S \cap T$ and also $T \supset S \cap T$, then $n \ge \#(S \cap T)$ and $m \ge \#(S \cap T)$, because the cardinal number of a subset is less than or equal to the cardinal number of the containing set. Thus $\min(m, n) \ge \#(S \cap T)$. Then the same example as in (a) shows that in some cases, $\min(m, n) = \#(S \cap T)$, so this is the largest number..

d) What is the smallest number of elements that $S \cap T$ can contain?

Answer: If S and T are disjoint, then $S \cap T$ is the empty set and has 0 elements.

e) Find a formula for the number of elements of $S \cup T$ in terms of the number of elements of S, T and $S \cap T$.

Answer: In class on Wednesday 1/25, we proved this formula:

$$\#(S \cup T) = \#S + \#T - \#(S \cap T)$$

Notice that most of (a) – (d) follows from this formula.

**Problem 4.4:** Gemignani, Section 6.3 # 3

Determine whether each of the statements is true or false. If a statement is true, supply a proof. If a statement I false, correct the statement and prove the corrected statement.

a) Any two subsets of the set of positive even integers contain the same number of elements.

This is clearly FALSE, since any two finite subsets with different cardinal numbers (for example {2} and {2, 4}) do not contain the same number of elements.

One corrected version that is true: Any two infinite subsets of the set of positive even integers contain the same number of elements.

Proof. The proof for Proposition 3 will also prove this statement.

b) Any two infinite subsets of the set of integers contain the same number of elements.

This is TRUE.

Proof: We will prove that any infinite subset of the set of integers is countable. Let S be such a subset and let $S_1$ be the set of elements in S that are $\geq 0$ and let $S_2$ be the set of elements of S that are $< 0$. Then $S = S_1 \cup S_2$ . By the reasoning of the proof of Proposition 3, we list the elements of $S_1$ in their natural order and this pairs up the elements of $S_1$ with N, the set of natural numbers. Likewise, list the elements of $S_2$ in their natural decreasing order to pair up the elements of this set with the numbers of N. To prove that S is countable, we need to pair up the elements of S with N. To do this, we pair the elements of S1 with the odd natural numbers and the elements of S2 with the even natural numbers: if an element s of S1 corresponds to n under the original pairing, under this new pairing it will correspond to 2n+1. If an element t of S2 corresponds to m, under the new pairing, let is correspond to 2m. This sets up a one-to-one correspondence between the elements of S and the elements of N, so S is countable.

c) If W is any subset of the set Z of positive integers then infinitely many subsets of Z contain the same number of elements as W.

This is FALSE as stated, because there is one exception.

One corrected version that is true: If W is any ***non-empty*** subset of the set Z of positive integers then infinitely many subsets of Z contain the same number of elements as W.

Proof. First, assume that W is a finite set. Then Z – M is an infinite set of integers.

Let n be any element of W (n exists, since W is not empty). Then for any $m \in Z - M$ let $W_m$ be the $(W - \{n\}) \cup \{m\}$. $W_m$ has the same number of elements as W, but each of these sets is different from the others, since $m \in W_m$ but $m \in W_n$ if $m \neq n$.

If W is infinite, W is countably infinite with the same number as N (by (b)). So W has the same number as Z – {n} for any $n \in Z$. And this is an infinite collection of subsets of Z.

d) An uncountable subset may contain a countable subset.

TRUE

Proof. Since the statement says "may" we only need an example. One example is the set of integers Z, which is a subset of the real numbers.

Comment: In fact, any infinite set S contains a countable subset. Just pick one element as $x_1$. Then pick $x_2 \in S - \{x_1\}$, $x_3 \in S - \{x_1, x_2\}$, etc. This process gives an infinite list of elements of S, since one runs out of elements only if S is finite. Then the set of $x_i$ is countable.

e) If S is a finite set, then S contains the same number of elements as the collection of subsets of S.

FALSE. For counterexample, see Problem 4.1

Corrected statement #1: If S is a finite set, S contains fewer elements than P(S), the set of subsets of S.

Proof: Let #S = n, and let $S = \{s_1, \ldots , s_n\}$. Then $\{\{\}, \{s_1\}, \ldots, \{s_n\}\}$ is a subset of P(S) that has n+1 elements. So $\#S = n < n+1 \leq \#P(S)$.

Corrected statement #2: If S is a finite set with n elements, then S has $2^n$ subsets.

Proof. This is a much stronger and better statement. We have seen this in examples and will prove it soon as an example of mathematical induction. If some students stated and proved this statement using induction (formally or informally), that is very good.

**Problem 4.5:** Gemignani, Section 6.5 # 2

Compute each of the following sums of cardinal numbers by means of the procedure described in Definition 6.4. Note: I am not sure how to create an over-bar with Word, so I am using underline instead. (Note: (a) (b) and (c) are set up in 3 different ways just to make it more interesting. Any of these methods will work for the others.)

a) $\underline{3} + \underline{1} = \underline{4}$

Proof: Let S = {1, 2, 3} and T = {4}.

Then $S \cup T = \{1, 2, 3, 4\}$ and $\#S = \underline{3}$, $\#T = \underline{1}$ and $\# S \cup T = \underline{4}$.

b) $\underline{6} + \underline{2} = \underline{8}$

Proof: Let S = {1, 2, 3, 4, 5, 6} and T = {a, b}.

Then S ∪ T = {1, 2, 3, 4, 5, 6, a, b} and #S = $\underline{6}$, #T = $\underline{2}$ and # S ∪ T = $\underline{8}$.

c) $\underline{9} + \underline{11} = \underline{20}$

Proof: Let S = {1, 2, 3, 4, 5, 6, 7, 8, 9} and T = {-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0}.

Then S ∪ T = {{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9} and #S = $\underline{9}$, #T = $\underline{11}$ and # S ∪ T = $\underline{20}$.

d) $\underline{3} + \underline{N} = \underline{N}$

Proof: Let S = {-2, -1, 0} and T = N. Then S ∪ T = {-2, -1, 0, 1, 2, 3, . . . } is countable, since it is an infinite subset of Z. So # S ∪ T = N. So $\underline{3} + \underline{N} = \underline{S} + \underline{N}$ = # S ∪ T = $\underline{N}$.

e) $\underline{3} + \underline{N} = \underline{N}$

Proof: Let S = {1, 2, 3, 4, 5, 6, 7} and T = {8, 9, 10, , . . . } is countable. Then S ∪ T = N. As before T is countably infinite, so 3 + $\underline{N}$ = #S + #T = # S ∪ T = $\underline{N.}$

**Problem 4.6**

Let S = {(2s, s - 2)| s ε **R**} and T = {(2t+2, t - 1)| t ε **R**}.

Show S = T or find an element of one that is not in the other.

In fact, S = T is True.

Proof. We need to show that every element of S is an element of T and vice-versa. Begin with an element p of S. Then by definition of S, p = (2s, s - 2) for some s ε **R.** Then we must show that there is a number t ε **R** so that p = (2t+2, t – 1).

So we write two equations relating s and t and solve.

2s = 2t+2 and s-2 = t-1. The first equation says s = t + 1, or t = s – 1. So this must be t if there is a t that works. Then we substitute into the formula for p to see whether we get the correct form for T: p = (2s, s - 2) = (2(t+1), {t+1) – 2) = (2t + 2, t – 1). Yes, this is T.

So we have shown that any p in S, p is also in T, by the correspondence s → t = s-1.

This is reversible. For any q in T, we have q = (2t + 2, t – 1) for some real number t. If we set s = t+1, so that t = s – 1, we have q = (2(s-1) + 2, (s-1) – 1) = (2s, s – 2), which is in S.

So S = T.

QED