Chapter 8: Theory Review: Solutions  
Math 308 F  
Spring 2015

1. If \( B = \{v_1, \ldots, v_k\} \) is an orthogonal basis for a subspace \( S \) of \( \mathbb{R}^n \), and \( v \) is any vector in \( S \), what is \( v_B \)?

Because \( B \) is an orthogonal basis, we know

\[
v = c_1v_1 + \cdots + c_kv_k
\]

where

\[
c_i = \frac{v \cdot v_i}{\|v_i\|^2}
\]

so

\[
v_B = \begin{bmatrix} \frac{v \cdot v_1}{\|v_1\|^2} \\ \vdots \\ \frac{v \cdot v_k}{\|v_k\|^2} \end{bmatrix}_B.
\]

2. Give an example of the following, or explain why such an example does not exist.

   (a) A nonzero vector \( u \) in both \( S \) and \( S^\perp \), where \( S \) is any subspace of \( \mathbb{R}^n \).

   This does not exist. If there were some vector \( u \) in both \( S \) and \( S^\perp \), then because \( u \) is in \( S \), we must have \( \text{proj}_S u = u \). But, because \( u \) is in \( S^\perp \), \( \text{proj}_S u = 0 \) (see problem 3(e)). So, \( u = \text{proj}_S u = 0 \), meaning the only vector \( u \) that is in both \( S \) and \( S^\perp \) is the zero vector.

   (b) An orthogonal basis for \( \mathbb{R}^3 \).

   The easiest example is the standard basis \( \{e_1, e_2, e_3\} \).

   (c) An orthogonal basis for the plane \( x + y + z = 0 \).

   There are two ways to do this problem: one way would be to find any basis for the given plane, then use the Gram-Schmidt Procedure to turn it into an orthogonal basis. Or, if you don’t want to do that, we could just find two linearly independent vectors (i.e. not multiples) in the plane that happen to be orthogonal. They would be linearly independent, so would span the plane, so would be an example of an orthogonal basis. For instance,

\[
\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}
\]

would be an orthogonal basis.
3. For each statement below, determine if it is True or False. Justify your answer. The justification is the most important part!

(a) If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in \( \mathbb{R}^n \), then \( \| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 \).

**FALSE.** For example, if \( \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \mathbf{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \), then \( \mathbf{u} + \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and \( 0 \neq 1 + 1 \). Note: the equation is true if and only if \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal.

(b) An orthogonal set of vectors is linearly independent.

**TRUE.** This is theorem 8.11 in the book.

(c) For any nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \), \( \text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{u}, \mathbf{v} - \text{proj}_u \mathbf{v}\} \).

**TRUE.** Regardless of what \( \mathbf{u} \) and \( \mathbf{v} \) are, proj\( \mathbf{u} \) is a scalar multiple of \( \mathbf{u} \). For simplicity, let’s not write the formula and just say proj\( \mathbf{u} \mathbf{v} = c \mathbf{v} \). Now, the question is asking: does \( \text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{u}, \mathbf{v} - c \mathbf{u}\} \)? Then answer is yes because, if \( \mathbf{x} \) is a linear combination of \( \mathbf{u} \) and \( \mathbf{v} \), then it can be expressed as a linear combination of \( \mathbf{u} \) and \( \mathbf{v} - c \mathbf{u} \), and vice versa.

(d) If \( \mathbf{u} \) is in the orthogonal complement of a subspace \( S \), then for any \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) in \( S \), \( \mathbf{u} \cdot (\mathbf{s}_1 + \mathbf{s}_2) = 0 \).

**TRUE.** If \( \mathbf{u} \) is in \( S^\perp \), by definition, \( \mathbf{u} \cdot \mathbf{s} = 0 \) for any vector \( \mathbf{s} \) in \( S \). But, because \( S \) is a subspace, if \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) are in \( S \), then \( \mathbf{s}_1 + \mathbf{s}_2 \) is in \( S \), so \( \mathbf{u} \cdot (\mathbf{s}_1 + \mathbf{s}_2) = 0 \).

(e) If \( S \) is a nonzero subspace of \( \mathbb{R}^n \), and \( \mathbf{u} \) is in \( S^\perp \), then proj\( \mathbf{S} \mathbf{u} = 0 \).

**TRUE.** Let \( \mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \) is an orthogonal basis for a subspace \( S \). By definition,

\[
\text{proj}_S \mathbf{u} = \text{proj}_{\mathbf{v}_1} \mathbf{u} + \cdots + \text{proj}_{\mathbf{v}_k} \mathbf{u}
\]

where

\[
\text{proj}_{\mathbf{v}_i} \mathbf{u} = \frac{\mathbf{v}_i \cdot \mathbf{u}}{\| \mathbf{v}_i \|^2} \mathbf{v}_i.
\]

But, if \( \mathbf{u} \) is in \( S^\perp \), then \( \mathbf{v}_i \cdot \mathbf{u} = 0 \) for each \( i \), so

\[
\text{proj}_S \mathbf{u} = 0.
\]

(f) If \( A \) is any matrix, and \( \mathbf{u} \) is in \( \text{null}(A) \), then \( \mathbf{u} \) is orthogonal to the rows of \( A \).

**TRUE.** If \( \mathbf{u} \) is in \( \text{null}(A) \), then, by definition, \( A \mathbf{u} = \mathbf{0} \). But, how do we actually compute \( A \mathbf{u} \) if \( a_1, \ldots, a_k \) are the rows of \( A \),

\[
A \mathbf{u} = \begin{bmatrix} a_1 \cdot \mathbf{u} \\ \vdots \\ a_k \cdot \mathbf{u} \end{bmatrix}.
\]
If $Au = 0$, this means $a_1 \cdot u = 0, \ldots, a_k \cdot u = 0$, i.e. $u$ is orthogonal to the rows of $A$.

(g) Given a linear system $Ax = b$, the corresponding normal equations always have a solution.

**TRUE.** This was the entire point of the derivation of the normal equations. If $\hat{y} = \text{proj}_{\text{col}(A)} y$, then $\hat{y}$ is in $\text{col}(A)$, so there is a solution, $Ax = \hat{y}$. But, $y - \hat{y}$ is in $(\text{col}(A))^\perp$, and $(\text{col}(A))^\perp = (\text{row}(A^T))^\perp = \text{null}(A^T)$ (do you remember why, given a matrix $B$, $\text{row}(B)^\perp = \text{null}(B)$? This has something to with (f) above.) Anyway, because $y - \hat{y}$ is in $\text{null}(A^T)$, by definition, $A^T(y - \hat{y}) = 0$. But, this says $A^T y = A^T \hat{y}$, so if $x$ is the solution to $Ax = \hat{y}$, it is also a solution to $A^T y = A^T A x$. The solution to $Ax = \hat{y}$ always exists, meaning the solution to $A^T y = A^T A x$ always exists.