Chapter 4: Theory Review
Math 308 F
Spring 2015

1. What is the definition of a subspace of $\mathbb{R}^n$?

2. Describe two ways to prove a subset $W$ of $\mathbb{R}^n$ is a subspace.

3. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Either prove that the subset if a subspace or provide a counterexample to one of the three conditions.

   (a) $W$ is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying $x_1 x_2 = x_3$

   (b) $W$ is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying $3x_1 + 4x_2 = x_3$

   (c) $W$ is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (d) $W$ is the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying $|x_1| = |x_2|$

   (e) $W$ is the line through the origin with direction vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
4. What is the definition of a basis for a subspace?

5. What is the definition of the dimension of a subspace?

6. For each of the subsets in problem 3, if it is a subspace, find a basis for it and the dimension of it.

7. Find two different bases for \( \text{col}(A) \), one consisting of vectors that are columns of \( A \), and one consisting of vectors that are NOT columns of \( A \), where \( A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -2 & 2 \end{bmatrix} \).

8. If \( \{a_1, \ldots, a_m\} \) is a basis for a subspace \( W \) of \( \mathbb{R}^n \), and \( A = [a_1 \ldots a_m] \), explain why the equation \( Ax = w \) always has a unique solution for any \( w \) in \( W \).

9. Give an example of the following, or explain why such an example does not exist.
   (a) A subspace of \( \mathbb{R}^n \) with no basis
   
   (b) A subspace of \( \mathbb{R}^3 \) with dimension 2
   
   (c) A subspace of \( \mathbb{R}^3 \) with dimension 3
   
   (d) A subspace of \( \mathbb{R}^3 \) with dimension 4
(e) An \( n \times m \) matrix \( A \), with \( m < n \), such that \( \text{rank}(A) = n \).

(f) An \( n \times m \) matrix \( A \), with \( m > n \), such that \( \text{rank}(A) = n \).

10. For each statement below, determine if it is True or False. *Justify your answer.* The justification is the most important part!

(a) If \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a linear transformation, and \( \ker(T) = \{0\} \), then \( m \leq n \).

(b) If \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is a linearly independent set of vectors in \( \mathbb{R}^m \) and \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a linear transformation such that \( \ker(T) = \{0\} \), then \( \{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\} \) is a linearly independent set.

(c) If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a linear transformation and \( \ker(T) = \{0\} \), then \( \text{range}(T) = \mathbb{R}^n \).

(d) If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a linear transformation and \( \ker(T) \neq \{0\} \), then \( T \) is not onto.

(e) Every subspace has a basis.

(f) There are infinitely many bases for any nonzero subspace.

(g) If \( \mathcal{B} \) is a basis for \( \mathbb{R}^n \), and \( W \) is a subspace of \( \mathbb{R}^n \), then some subset of the vectors in \( \mathcal{B} \) form a basis for \( W \).

(h) If \( W \) is a subspace of \( \mathbb{R}^n \) and \( \mathcal{B} \) is a basis for \( W \), then \( \mathcal{B} \) can be extended to a basis for \( \mathbb{R}^n \).

(i) If \( A \) is a \( 3 \times 5 \) matrix, then the maximum value of \( \text{rank}(A) \) is 5.

(j) If \( A \) is an \( n \times m \) matrix and \( n < m \), then \( \text{nullity}(A) \) must be greater than 0.

(k) If \( A \) is a nonsingular \( n \times n \) matrix, then the columns of \( A \) form a basis for \( \mathbb{R}^n \).

(l) If \( A \) is a singular \( n \times n \) matrix, then \( \text{nullity}(A) \neq 0 \).