1. What two properties must a function $T: \mathbb{R}^m \to \mathbb{R}^n$ satisfy to be a linear transformation?

2. If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, what is the definition of onto? of one-to-one? Explain how these definitions relate to the matrix $A$ such that $T(x) = Ax$.

3. Three functions are given below. Determine if they are linear transformations (by checking that they satisfy the two properties above) and, if a function is a linear transformation, determine if it is onto or one-to-one (or both).

   (a) $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by
   
   $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 + 3 \end{bmatrix}$$

   (b) $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by
   
   $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_3 - 4x_2 \end{bmatrix}$$

   (c) Let $y$ be a fixed vector in $\mathbb{R}^3$, then let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be given by
   
   $$T(x) = (y \cdot x)y$$
4. Give an example of the following, or explain why such an example does not exist.

   (a) A linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) such that \( T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \)

   (b) A linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) such that \( T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \) and \( T \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)

   (c) A linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) such that \( T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \) and \( T \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)

   (d) Non-square matrices \( A \) and \( B \) such that \( AB = I \)

   (e) Non-square matrices \( A \) and \( B \) such that \( AB = I \) and \( BA = I \)

   (f) An \( n \times n \) matrix \( A \) with no zero entries such that \( A \) is nonsingular

   (g) An \( n \times n \) matrix \( A \) with no zero entries such that \( A \) is singular

   (h) A singular matrix \( A \) whose columns are linearly independent

   (i) A square matrix \( A \) such that \( A^{-1} = A^T \)
5. For each statement below, determine if it is True or False. *Justify your answer.* The justification is the most important part!

(a) If $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation, then $T(0) = 0$.

(b) If $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation, then the only vector $x$ such that $T(x) = 0$ is $x = 0$.

(c) If $\{v_1, v_2, v_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^m$ and $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation, then $\{T(v_1), T(v_2), T(v_3)\}$ is a linearly dependent set.

(d) If $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors in $\mathbb{R}^m$ and $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation, then $\{T(v_1), T(v_2), T(v_3)\}$ is a linearly independent set.

(e) For any matrix $A$, there is a matrix $B$ such that $AB = A$.

(f) For any matrix $A$, there is a matrix $B$ such that $AB = I$.

(g) If $AB = AC$ and $A$ is not the zero matrix, then $B = C$.

(h) If $B$ is a matrix whose columns are linearly dependent, then the columns of $AB$ are linearly dependent (where $A$ is any matrix such that the product $AB$ is defined).

(i) If $B$ is a matrix whose columns are linearly independent, then the columns of $AB$ are linearly independent (where $A$ is any matrix such that the product $AB$ is defined).

(j) If $A$ is not the zero matrix, then $A$ has an inverse.

(k) There is a nonsingular matrix $A$ such that $A^2 = 0$ (the zero matrix).

(l) There is a nonsingular matrix $A$ such that $A^2 = A$.

(m) If $A$ and $B$ are nonsingular $(n \times n)$ matrices, then $A + B$ is also nonsingular.

(n) If $A$ and $B$ are nonsingular $(n \times n)$ matrices, then $AB$ is also nonsingular.