Do not open this exam until you are told to begin. You will have 50 minutes for the exam.

Check that you have a complete exam. There are 4 questions for a total of 50 points.

You are allowed to have one single sided, handwritten note sheet. Calculators are not allowed.

Cheating will result in a zero and be reported to the Dean’s Academic Conduct Committee.

Show all your work. With the exception of True/False questions, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

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1. (18 points) True/False and short answer. For these questions, you are not required to show any work.

(a) A system of equations with more variables than equations cannot have a unique solution.
   √ True  ○ False

(b) If \( m < n \), a set of \( m \) vectors in \( \mathbb{R}^n \) cannot span \( \mathbb{R}^n \).
   √ True  ○ False

(c) If \( m < n \), any set of \( m \) vectors in \( \mathbb{R}^n \) is linearly independent.
   ○ True  √ False

(d) If the equation \( Ax = 0 \) has a unique solution, then the columns of \( A \) are linearly independent.
   √ True  ○ False

(e) If \( \{u_1, u_2, u_3\} \) is linearly dependent, then \( \{u_1, u_2, u_3, u_4\} \) is linearly dependent (where all vectors have the same dimension).
   √ True  ○ False

(f) If \( u_4 \) is not a linear combination of \( \{u_1, u_2, u_3\} \), then \( \{u_1, u_2, u_3, u_4\} \) is linearly independent (where all vectors have the same dimension).
   ○ True  √ False

(g) Give an example of a linear system with more variables than equations that has no solution.

**Solution:** One such example is the system

\[
\begin{align*}
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_2 + x_3 &= 1
\end{align*}
\]

(h) Give an example of three distinct nonzero vectors in \( \mathbb{R}^2 \) that don’t span \( \mathbb{R}^2 \).

**Solution:** One such example is

\[
\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}
\]

(i) Give an example of two vectors \( u_1 \) and \( u_2 \) in \( \mathbb{R}^3 \) such that \( \text{span}\{u_1, u_2\} \) is the set of all vectors \( v \) of the form \( v = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \).

**Solution:** One example is

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
2. Consider the linear system

\[
\begin{align*}
    x_1 - 2x_3 + x_4 &= 0 \\
    x_1 + x_2 - 2x_3 &= 0 \\
    x_1 - 2x_2 - 2x_3 + 3x_4 &= 0
\end{align*}
\]

(a) (5 points) Solve the system and write your answer in vector form.

\textbf{Solution:} The matrix

\[
\begin{bmatrix}
    1 & 0 & -2 & 1 & 0 \\
    1 & 1 & -2 & 0 & 0 \\
    1 & -2 & -2 & 3 & 0
\end{bmatrix}
\]

reduces to

\[
\begin{bmatrix}
    1 & 0 & -2 & 1 & 0 \\
    0 & 1 & 0 & -1 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The variables \(x_3\) and \(x_4\) are free, so assigning the free parameters \(x_3 = s_1\) and \(x_4 = s_2\), the solution is

\[
\begin{align*}
    x_1 &= 2s_1 - s_2 \\
    x_2 &= s_2 \\
    x_3 &= s_1 \\
    x_4 &= s_2
\end{align*}
\]

where \(s_1\) and \(s_2\) are any real numbers. In vector form, the solutions are

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = s_1 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

where \(s_1\) and \(s_2\) are any real numbers.

(b) (3 points) Find vectors \(u_1\) and \(u_2\) such that the solution set is given by \(\text{span}\{u_1, u_2\}\). (Hint: your answer to (a) may be helpful).

\textbf{Solution:} The span of a set of vectors is all linear combinations of those vectors, and our answer to (a) tells us that every solution can be written as a sum of

\[
\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

so we can set

\[
\begin{align*}
    u_1 &= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]
3. Consider the vector equation
\[
\begin{bmatrix}
1 & -1 & -2 \\
1 & 1 & -4 \\
2 & -1 & a
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
2 \\
b
\end{bmatrix}
\]

(a) (4 points) For what values of \(a\) and \(b\) does the equation have no solution?

**Solution:** To solve the equation, we reduce
\[
\begin{bmatrix}
1 & -1 & -2 & | & 0 \\
1 & 1 & -4 & | & 2 \\
2 & -1 & a & | & b
\end{bmatrix}
\]
to get
\[
\begin{bmatrix}
1 & 0 & -3 & | & 1 \\
0 & 1 & -1 & | & 1 \\
0 & 0 & a+5 & | & b-1
\end{bmatrix}.
\]
The only time this has no solution is when \(a + 5 = 0\) and \(b - 1 \neq 0\), or \(a = -5\) and \(b \neq 1\) (so we get a row of zeros with a nonzero entry in the last column).

(b) (4 points) For what values of \(a\) and \(b\) does the equation have infinitely many solutions?

**Solution:** Using the reduced matrix from part (a), this has infinitely many solutions whenever the bottom row is entirely zero, so \(a = -5\) and \(b = 1\).

(c) (4 points) Give an example of \(a\) and \(b\) where the equation has exactly one solution, and solve for \((x_1, x_2, x_3)\) in that case.

**Solution:** One example: \(a = -4\) and \(b = 1\). Then, the matrix becomes
\[
\begin{bmatrix}
1 & 0 & -3 & | & 1 \\
0 & 1 & -1 & | & 1 \\
0 & 0 & 1 & | & 0
\end{bmatrix}
\]
which reduces to
\[
\begin{bmatrix}
1 & 0 & 0 & | & 1 \\
0 & 1 & 0 & | & 1 \\
0 & 0 & 1 & | & 0
\end{bmatrix}
\]
so \((x_1, x_2, x_3) = (1, 1, 0)\).
4. Consider the vectors
\[ u_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} \]

(a) (4 points) Show that the vectors \( \{u_1, u_2, u_3\} \) do NOT span \( \mathbb{R}^3 \) and give an example of a vector \( v \) that is not in their span.

**Solution:** Begin by reducing
\[
\begin{bmatrix}
1 & -1 & -2 & | a \\
-1 & 2 & 3 & | b \\
-2 & 2 & 4 & | c
\end{bmatrix}
\]
to
\[
\begin{bmatrix}
1 & 0 & -1 & | 2a + b \\
0 & 1 & 1 & | a + b \\
0 & 0 & 1 & | 2a + c
\end{bmatrix}.
\]
This tells us the only vectors \( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) that are in \( \text{span}\{u_1, u_2, u_3\} \) are ones such that
\[ 2a + c = 0. \]
To find a vector not in the span, we can choose any \( v \) that does not satisfy this, such as
\[ v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]

(b) (4 points) Are the vectors \( \{u_1, u_2, u_3, v\} \) linearly independent? (Note \( v \) should be the vector you found in part (a).) Justify your answer.

**Solution:** These are not linearly independent because \( \{u_1, u_2, u_3\} \) is not a linearly independent set, which we know because the Big Theorem tells us \( \{u_1, u_2, u_3\} \) spans \( \mathbb{R}^3 \) if and only if \( \{u_1, u_2, u_3\} \) is linearly independent. (Note: we could also show this by reducing a matrix.)

(c) (4 points) Do the vectors \( \{u_1, u_2, u_3, v\} \) span \( \mathbb{R}^3 \)? Justify your answer.

**Solution:** These vectors do span \( \mathbb{R}^3 \). We could show this by reducing a matrix or noting that \( \{u_1, u_2\} \) is linearly independent because \( u_1 \) and \( u_2 \) are not multiples of each other, and \( v \) is not a linear combination of \( \{u_1, u_2\} \) (which we know from part (a)). Therefore, \( \{u_1, u_2, v\} \) is linearly independent, so by the Big Theorem, spans \( \mathbb{R}^3 \). Therefore, \( \{u_1, u_2, u_3, v\} \) spans \( \mathbb{R}^3 \).