15.1 Volumes and Double Integrals

Consider a function $f$ of 2 variables defined on a closed rectangle $R = [a, b] \times [c, d]$. Assume $f(x, y) \geq 0$ for now.

The graph of $f$ is a surface with equation $z = f(x, y)$.

Let $S$ be the solid that lies above $R$ and under the graph of $f$, i.e., $S = \{ (x, y, z) \in \mathbb{R}^3 | 0 \leq z \leq f(x, y), (x, y) \in R \}$.

Goal: Find the volume of $S$.

$V = \iint_R f(x, y) \, dA.$

(This is the limit of Riemann sums as the areas of rectangles $\to 0$.)

Average Value.

Define the average value of a function $f$ of 2 variables defined on a rectangle $R$ as

$$\bar{f}_{av} = \frac{1}{\text{(area of } R)} \iint_R f(x, y) \, dA.$$
Computing Double Integrals with Iterated Integrals.

We express the Double Integral as an iterated integral which is then evaluated by calculating two single integrals.

1. Compute \( A(x) = \int_{c}^{d} f(x, y) \, dy \) (Treat \( x \) as a constant).

2. Then compute \( \int_{a}^{b} A(x) \, dx \).

\[ \int_{a}^{b} \left[ \int_{c}^{d} f(x, y) \, dy \right] \, dx \]

We usually omit the brackets to write:

\[ \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \]

Nothing special about the order. We can instead compute:

\[ \int_{a}^{b} \left[ \int_{a}^{b} f(x, y) \, dx \right] \, dy \]

This follows from Fubini's Theorem:

If \( f \) is continuous on \( R = [a, b] \times [c, d] \) then

\[ \int_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy \]

(We do not need to assume \( f \geq 0 \) on \( R \). The theorem works for any 'reasonable' function.)
Exercises:

15.2.3 \int \int \int (6x^2 y - 2x) \, dy \, dx.

15.2.14 \int \int \int \sqrt{s + t} \, ds \, dt.

Example 4: Find the volume of the solid \( S \) that is bounded by the elliptic paraboloid \( x^2 + 2y^2 + z = 16 \), the plane \( x=2 \), \( y=2 \) and the three coordinate planes.

Note that \( S \) is enclosed by \( z = 16 - x^2 - 2y^2 \) and \( z = 0 \), and lies above \( [0,2]^2 \times [0,2] \).

\[ 15.2.29 \] Find the volume of solid enclosed by \( z = x \sec^2 y \), \( z = 0 \), \( x = 0 \), \( x = 2 \), \( y = 0 \) \& \( y = \pi/4 \).

\( S \) is bounded by \( z = x \sec^2 y \) and \( z = 0 \) and lies above \( [0,2] \times [0,\pi/4] \).

\[ 15.2.30 \] Find the volume of solid in the first octant bounded by \( z = 16 - x^2 \) and plane \( y = 5 \).
In particular this is the case for the following 2 types of regions.

**Type 1.**

\[
y = g_1(x) \quad \text{and} \quad y = g_2(x)
\]

[Some examples of Type 1 region]

\[ D = \{ (x, y) : a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x) \} \]

i.e. \( D \) lies b/w the graphs of 2 continuous functions of \( x \).

To evaluate \( \iint_D f(x, y) \, dx \, dy \) where \( D \) is a type 1 region, we use:

\[
\iint_D f(x, y) \, dx \, dy = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \right] \, dx
\]

**Type 2.**

\[ D = \{ (x, y) : c \leq y \leq d, \ h_1(y) \leq x \leq h_2(y) \} \]

i.e. \( D \) lies b/w graphs of 2 continuous functions of \( y \).
* We can similarly show that
\[ \int \int_D f(x, y) \, dA = \int \int_{D_{1(y)}} f(x, y) \, dx \, dy \]
where \( D \) is a type II region.

**Exercises:**

**Example 1:** Evaluate \( \int \int_D (x+2y) \, dA \) where \( D \) is the region bounded by \( y = 2x^2 \) and \( y = 1+x^2 \).

**Example 2:** Find the volume of solid that lies under \( z = x^2 + y^2 \) above the region \( D \) in the \( xy \)-plane bounded by \( y = 2x \) and \( y = x^2 \).

**Example 4:** Find the volume of the tetrahedron bounded by the planes \( x+2y+z = 2 \), \( x = 2y \), \( x = 0 \) and \( z = 0 \).

What is the region of integration \( D \)?

**Example 5:** Evaluate \( \int \int_D \sin(y^2) \, dy \, dx \)

**15-3-21:** \( \int \int_D (2x-y) \, dA \) where \( D \) is bounded by the circle with origin as center and radius 2.
15.3.27) Find the volume of the solid bounded by
\[ y^2 + z^2 = 4 \] and the planes \( x = 2y, x = 0, \)
in the first octant.

15.3.47) Change the order of integration:
\[ \int \int \int f(x,y) \, dy \, dx. \]

15.3.49) Evaluate:
\[ \int_{0}^{3} \int_{0}^{3} e^{x^2} \, dy \, dx. \]

\[ \int \int \text{Double Integrals in Polar Coordinates} \]

If we wish to evaluate a double integral \( \int \int f(x,y) \, dA \)
where \( R \) is either
\[ \text{or} \]
then \( \int \int f(x,y) \, dA \) is evaluated more easily
by switching to polar coordinates.
* Recall that \((r, \theta)\) is related to \((x, y)\) by
\[
\begin{align*}
(x, y) & \quad \text{(Cartesian coordinates)} \\
(r, \theta) & \quad \text{(Polar coordinates)}
\end{align*}
\]

by \(x = r \cos \theta\), \(y = r \sin \theta\).

The regions in the previous page are typical examples of polar rectangle

\[
R = \{ (r, \theta) : a \leq r \leq b, \quad a \leq \theta \leq \beta \}
\]

B If \(f\) is integrable on a polar rectangle \(R\), then

\[
\int_R f(x, y) \, dA = \int_a^b \int_a^b f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta
\]

Be careful not to forget the additional factor \(r\) in the above formula!

A good thumb rule is to sketch the region of integration to decide whether to convert the problem to a polar framework.
Exercises:

15.4.11 Evaluate
\[ \iint_D e^{-(x^2+y^2)} \, dA \] where \( D \) is the region bounded by the semi-circles \( x = \sqrt{4-y^2} \) and \( y = 4-x^2 \).

15.4.15 Use double integral to evaluate the area of the region inside \((x-1)^2 + y^2 = 1\) and outside \(x^2 + y^2 = 1\).

15.4.25 Find the volume of the solid above \( z = \sqrt{x^2+y^2} \) and below \( x^2 + y^2 + z^2 = 1 \).

15.4.39 Use polar coordinates to combine the sum
\[ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sqrt{2}}^{2} r \sin \theta \, dr \, d\theta \]
and
\[ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sqrt{2}}^{2} r^2 \sin \theta \cos \theta \, dr \, d\theta \]
into one double integral. Then evaluate the double integral.