Survey on Dieudonné theory

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Introduction

Dieudonné theory is named after the French mathematician *Jean Dieudonné* (1906-1992), and was developed by him to study formal groups. Today his work has influenced stable homotopy theory, modular representation theory, and the theory of abelian varieties over fields of positive characteristic. It is fair to say that anyone interested in positive characteristic phenomena should hear about Dieudonné theory at some point in their lifetime.

This document is an expository paper on Dieudonné theory and is written as accompanying notes for lectures given by the author as part of a reading course with Professor Julia Pevtsova at the University of Washington in the spring of 2025. We hope to provide a concise English reference for Dieudonné theory which is understandable to readers with some background in group schemes, e.g. from Waterhouse [7]. Our primary reference on Dieudonné theory is Demazure and Gabriel [5] and we will frequently translate directly from the French, changing only basic conventions and notations.

The author's primary interest is in the representation theory of finite group schemes over an algebraically closed field of positive characteristic. The basic application of Dieudonné theory that we will cover is philosophical: we hope to augment the way we understand Cartier duality for finite abelian group schemes. Representations over a given finite abelian group scheme coincide with sheaves over the Cartier dual, which are often more practical to understand. In most cases, a detailed formulation of the coordinate algebra of the Cartier dual still proves difficult. Finite abelian group schemes have a nice description as *Dieudonné modules* which aide in such a formulation.

Applications outside of representation theory not covered: In stable homotopy theory, a stack classifying formal groups is associated to the Thom spectrum representing complex cobordisms. Dieudonné theory gives a concrete way of talking about the heights of formal groups in positive characteristic which in turn offers a stratification of the associated stack. On another note, Hodge theory and analytical techniques are instrumental to the study of abelian varieties over the complex numbers, but no directly analogous theories do the trick for studying abelian varieties over fields of positive characteristic. Dieudonné theory offers a different avenue for understanding abelian varieties which is unique to positive characteristic base fields.

Contents

1 Recollections

First we will work over an arbitrary field k. Let A be a commutative k-algebra. We denote by $\operatorname{Spec} A$ the affine scheme over k. For most purposes, affine schemes over k are thought of as representable functors from commutative k-algebras to sets, as in Waterhouse [7], but it is greatly convenient to think of them also as locally ringed spaces as in Hartshorne [6]. If X is an affine scheme, we denote by $\mathcal{O}(X)$ the coordinate algebra, also called the representing algebra for the functor of points. In either description, we have equivalences of categories

$$\begin{tikzcd}
\{\text{Commutative } k\text{-algebras}\}^{\mathrm{op}} \arrow[r, "\sim"] & \{\text{Affine schemes over } k\} \\
\{\text{Commutative Hopf algebras}\}^{\mathrm{op}} \arrow[r, "\sim"] \arrow[u, hook] & \{\text{Affine group schemes}\} \arrow[u, hook] \\
\{\text{Bicommutative Hopf algebras}\}^{\mathrm{op}} \arrow[r, "\sim"] \arrow[u, hook] & \{\text{Affine abelian group schemes}\}, \arrow[u, hook]
\end{tikzcd}$$

where each horizontal arrow is the equivalence given by the functor $\operatorname{Spec}(-)$, with inverse $\mathcal{O}(-)$. On the purely algebraic side, whenever A is an associative algebra over k, we can dualize to get $A^* = \operatorname{Hom}_k(A, k)$, a coassociative coalgebra. Likewise we can dualize coalgebras to get algebras in such a manner that we have an isomorphism $(A^*)^* \cong A$ of algebras whenever A is finite dimensional over k. In this process, we have A^* is cocommutative whenever A is commutative and vice versa. As such, finite dimensional bicommutative Hopf algebras have a bicommutative Hopf algebra dual. Translated into geometry, this is known as *Cartier duality*. That is, if G is a finite abelian group scheme (i.e. an affine group scheme with a finite dimensional coordinate algebra), we denote by $G^\sharp$ the *Cartier dual* of G, which is given by $G^\sharp = \operatorname{Spec}(\mathcal{O}(G)^*)$. Cartier duality respects direct products of groups, as linear duals respect tensor products of finite vector spaces.

Example 1.0.1. The finite commutative algebra $\mathcal{O}(\mu_n) = \frac{k[x]}{x^n-1}$ has the structure of a bicommutative Hopf algebra with comultiplication $x \mapsto x \otimes x$. Thus $\mathcal{O}(\mu_n)$ is the coordinate algebra for an abelian group scheme which we call μ_n as notation suggests.

The finite group $\mathbb{Z}/n$ is a discrete topological group, and after identifying each point with $\operatorname{Spec} k$, is endowed with the structure of a finite group scheme over k.

We have that $\mathbb{Z}/n \cong \mu_n^\sharp$ as group schemes, for any n, and over any base field k. Whenever the polynomial $x^n - 1$ is separable and splits over k (e.g. k is algebraically closed of characteristic 0) we have also that $\mathbb{Z}/n \cong \mu_n$ as group schemes, and hence μ_n is self-dual. If $n = p^r$ and p is the characteristic of k, the group schemes $\mathbb{Z}/p^r$ and μ_{p^r} are drastically different, even as schemes: $\mathbb{Z}/p^r$ is reduced but not connected, and μ_{p^r} is connected but not reduced.

Example 1.0.2. Let k be a field of characteristic $p > 0$. The finite commutative algebra $\mathcal{O}(\alpha_{p^r}) = \frac{k[t]}{t^{p^r}}$ has the structure of a bicommutative Hopf algebra with comultiplication $t \mapsto t \otimes 1 + 1 \otimes t$. Thus $\mathcal{O}(\alpha_{p^r})$ is the coordinate algebra for an abelian group scheme which we call α_{p^r} as notation suggests.

For $r = 1$, we have $\alpha_p^\sharp \cong \alpha_p$, so we have another example of a self-dual finite abelian group scheme. For larger values of r, α_{p^r} is never self dual. In fact, the coordinate algebra for $\alpha_{p^r}^\sharp$, i.e. the dual Hopf algebra to $\mathcal{O}(\alpha_{p^r})$, is given by

$$\mathcal{O}(\alpha_{p^r}^\sharp) \cong \frac{k[t_0, \dots, t_{r-1}]}{(t_0^p, \dots, t_{r-1}^p),}.$$

where each t_i is the dual element to t^{p^i} with respect to the basis of homogeneous monomials in $\mathcal{O}(\alpha_{p^r})$. We have, given that α_p is a closed subgroup of α_p, that the comultiplication takes $t_0 \mapsto t_0 \otimes 1 + 1 \otimes t_0$. But for for $i > 0$ the comultiplication formula on t_i is not easy to write down, and is in and of itself a motivation for Dieudonné theory.

Note 1.0.3. Group schemes over a field of characteristic 0 are étale. For this reason the structure theory of finite group schemes is comparitively less interesting in this case than in the alternative: over an algebraically closed field of characteristic 0, finite group schemes are just finite groups, and for non algebraically closed fields it is reasonable to make a statement like 'finite group schemes are finite groups plus Galois theory' to make a distinction between e.g. μ_n and $\mathbb{Z}/n$. Non étale group schemes in characteristic $p > 0$ include both α_{p^r} and μ_{p^r}: neither are reduced (in fact they are isomorphic to one another as schemes, just not as groups). It is still true over arbitrary fields that finite *reduced* group schemes are étale, and so the structure theory of finite reduced group schemes over k may still be described as 'finite groups plus Galois theory'.

1.1 The Frobenius homomorphism

Let k be a field of characteristic $p > 0$. For any affine scheme X over k (it is not really necessary for X to be affine), we define the *absolute Frobenius* morphism $F_X : X \to X$ to correspond to the morphism $\mathcal{O}(X) \to \mathcal{O}(X)$ given by $\sigma \mapsto \sigma^p$ for each coordinate function $\sigma \in \mathcal{O}(X)$. By abuse of notation in the case of $X = \operatorname{Spec} k$ we will call F_X by F_k and also refer to the map of rings $k \to k$ as the absolute Frobenius F_k when the context is clear. Notice, F_X is not a morphism of schemes over k unless $k = \mathbb{F}_p$. Instead we have a commutative diagram of schemes over $\mathbb{Z}$

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

for any morphism of schemes $f : X \to Y$ over k. In the case when $f : X \to \operatorname{Spec} k$ is the structure morphism, we see that F_X is instead *skew*-linear over k with respect to the Frobenius endomorphism on k. Denote by $X^{(p)}$ the base change $\operatorname{Spec} k \times_{F_k} X$ along the absolute Frobenius $F_k : \operatorname{Spec} k \to \operatorname{Spec} k$. The *relative*

Frobenius morphism $F_{X/k} : X \to X^{(p)}$ is defined via the base change property

$$\begin{tikzcd}
X \arrow[r, dotted, "F_{X/k}"] \arrow[rr, bend left, "F_X"] \arrow[dr] & X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "F_k"'] & \operatorname{Spec} k
\end{tikzcd} .$$

Similarly we denote $X^{(p)^r}$ the base change along the rth iteration F_K^r, and $F_{X/k}^r : X \to X^{(p)^r}$ the rth relative Frobenius.

Proposition 1.1.1. Let G be any (affine) group scheme over k. Then $G^{(p)}$ is also a group scheme over k, and $F_{G/k} : G \to G^{(p)}$ is a homomorphism of group schemes over k.

One may ask how to characterize the Cartier dual of the relative Frobenius homomorphism on a finite abelian group scheme over k. That is, when G is a finite abelian group scheme, then since the Cartier dual $G^\sharp$ is also some group scheme, we have the relative Frobenius $F_{G^\sharp/k} : G^\sharp \to (G^\sharp)^{(p)}$ is a homomorphism of group schemes. Now dualizing back we have some homomorphism of group schemes

$$V_G : G^{(p)} \to G$$

(after identifying $((G^\sharp)^{(p)})^\sharp \cong G^{(p)}$). The question is then what is a larger context that our morphism V_G can be defined within? For arbitrary affine group schemes G (again affine is not really a necessary assumption, but it turns out, over bases other than a field, flatness becomes necessary), we can define a *Verschiebung* homomorphism $V_G : G^{(p)} \to G$ which coincides in the case of finite abelian group schemes to the Cartier dual of the relative Frobenius defined above.

1.2 The Verschiebung homomorphism

Verschiebung is German for *shift* (and is therefore capitalized, being a noun). Demazure and Gabriel [5] call the Verschiebung homomorphism by its German-to-French translation *décalage*, but the convention we'll follow in English is to keep the original German word. What precisely is 'shifting' in the Verschiebung homomorphism will become apparent after dealing with Witt vectors. Our treatment of the Verschiebung homomorphism below is following [5, IV, §3, n° 4].

1.2.1. The n-fold tensor power of a vector space M is denoted by $\otimes^n M$. The symmetric group $\mathfrak{S}_n$ acts linearly on $\otimes^n M$ by permutation on simple tensors. The $\mathfrak{S}_n$-invariants $(\otimes^n M)^{\mathfrak{S}_n}$ are denoted by $TS^n(M)$. Denote by $s : \otimes^n M \to TS^n(M)$ the linear map given by

$$s(m_1 \otimes \ldots m_n) = \sum_{\sigma \in \mathfrak{S}_n} m_{\sigma(1)} \otimes \ldots m_{\sigma(n)}.$$

Denote $M^{(p)} = k \otimes_{F_k} M$ the extension of scalars along $F_k : k \to k$.

Lemma 1.2.2. Take $n = p$. The map $M^{(p)} \to TS^p(M)/s(\otimes^p M)$ defined by mapping m to the image of $m \otimes \cdots \otimes m$ is a linear isomorphism.

Proof. It's clear that the map is linear. Fix a basis $(m_i)_{i\in I}$ for the vector space M. For each $\varphi \in I^p$, denote $m_\varphi = m_{\varphi_1} \otimes \cdots \otimes m_{\varphi_p}$, and denote $\omega(\varphi)$ the orbit of φ under the permutation action of $\mathfrak{S}_p$ on I^p. Whenever ω is an orbit of $\mathfrak{S}_p$ in I^p, let $m_\omega = \sum_{\varphi\in\omega} m_\varphi$. Then the elements m_ω form a basis for $TS^p(M)$, and $s(m_\varphi) = n_\varphi m_{\omega(\varphi)}$ for each $\varphi \in I^p$, where n_φ is the order of the centralizer of φ in $\mathfrak{S}_p$. Therefore $TS^p(M)/(s(\otimes^p))$ has a basis of the images of elements m_φ where $\varphi = (i, \dots i)$ for some constant index $i \in I$, i.e. the image under our map of basis elements m_i. □

1.2.3. Let R be a commutative k-algebra and $X = \operatorname{Spec} R$. The symmetric group $\mathfrak{S}_p$ acts on $X^p = \operatorname{Spec}(\otimes^p R)$ by permutation. We denote $S^p X = \operatorname{Spec} TS^p(R)$ where $TS^p(R) \subset \otimes^p R$ is the invariant subalgebra, and the map of schemes $X^p \to S^p X$ is identified as the canonical quotient $X^p \to X^p/\mathfrak{S}_p$, satisfying the usual universal property.

Notice the subspace $s(\otimes^p R)$ is an ideal in the algebra $TS^p(R)$: if $v \in TS^p(R)$ then $s(uv) = s(u)v$ for any $u \in \otimes^p R$. Further, $R^{(p)}$ is a commutative algebra, with $X^{(p)} = \operatorname{Spec} R^{(p)}$ by the prior definition for schemes, and the linear isomorphism $R^{(p)} \to TS^p(R)$ is indeed an isomorphism of algebras. Therefore for any affine scheme X over k we have a closed immersion $\mathfrak{i}_X : X^{(p)} \to S^p X$.

Lemma 1.2.4. If X is an affine scheme over k, then the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{diag}"] \arrow[d, "F_X"'] & X^p \arrow[d] \\ X^{(p)} \arrow[r, "\mathfrak{i}_X"] & S^p X \end{tikzcd}$$

commutes. Here diag is the diagonal and $X^p \to S^p X$ is the canonical projection explained above.

1.2.5. Let X be an affine scheme and G an affine *abelian* group scheme written additively (both over k) Let $f : X \to G$ be any morphism over k. Denote by $\sigma_p : G^p \to G$ the p-fold summation for the group G. Since G is abelian, $f_p = \sigma_p \circ f^p : X^p \to G$ is $\mathfrak{S}_p$-symmetric in the sense that it descends to a morphism from the quotient $X^p/\mathfrak{S}_p = S^p X$ we'll call $\overline{f}_p : S^p \to G$.

Define $f^V = \overline{f}_p \circ \mathfrak{i}_X : X^{(p)} \to G$. Now we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{diag}"] \arrow[d, "F_X"'] & X^p \arrow[r, "f^p"] \arrow[d] & G^p \arrow[d, "\pi_p"] \\ X^{(p)} \arrow[r, "\mathfrak{i}_X"] \arrow[rr, bend right, "f^V"'] & S^p X \arrow[r, "\overline{f}_p"] & G. \end{tikzcd}$$

We conclude that for any morphism $f : X \to G$, thought of as a point in the $\mathbb{Z}$-module $G(X)$, we have that $f^V \circ F_X$ agrees with multiplying f by p.

For the identity morphism $\mathrm{id} : G \to G$, we call $V_G = (\mathrm{id})^V : G^{(p)} \to G$ the *Verschiebung* homomorphism. Indeed, V_G is a homomorphism of group schemes.

Proposition 1.2.6. Let G be a finite abelian group scheme over k, and let $V' : G^{(p)} \to G$ be the homomorphism defined as the Cartier dual to the relative Frobenius $F_{G^\sharp/k}$ as before. Then $V' = V_G$, the Verschiebung homomorphism.

Proof. Let $R = \mathcal{O}(G)$ and hence $R^* = \mathcal{O}(G^\sharp)$. Then the Frobenius $F_{G^\sharp/k}$ at the level of algebras has a linear decomposition as

$$(R^*)^{(p)} \to S^p R^* \to R^*,$$

where the first map is the linear map into the pth symmetric power taking x to $x \otimes \cdots \otimes x$, and the second map is p-fold multiplication $\otimes^p R^* \to R^*$, descended to the quotient (since R^* is a commutative algebra, G being abelian). Dualizing we then have V', at the level of algebras, has a linear decomposition as

$$R \to TS^p(R) \to R^{(p)},$$

identifying the dual as $(S^p R^*)^* = (\otimes^p R^*/\mathfrak{S}_p)^* = (\otimes^p R)^{\mathfrak{S}_p} = TS^p(R)$. The linear decomposition of the Frobenius $F_{G^\sharp}$ was not a diagram of algebras (indeed, what is the algebra structure on $S^p A$ given a finite algebra A?) and hence it wouldn't make sense to take Spec. It was however a diagram of coalgebras, and hence dualizing gets us a diagram of algebras decomposing V'. It is easily verified, after dualizing, that $R \to TS^p(R)$ is the coordinate algebra map for $\overline{f}_p$, and that $TS^p(R) \to R^{(p)}$ is the quotient of Lemma 1.2.2, i.e. the coordinate algebra map for our closed immersion $\mathfrak{i}_G$. Hence taking Spec, our definition of V' has the same decomposition as that which defines V_G. $\square$

2 Witt Calculus

Ernst Witt (1911 - 1991) was among the last students of Emmy Noether during her time in Göttingen before the expulsion of all Jews from German universities by the Nazi regime in 1933. Witt, like his contemporary and friend Oswalt Teichmüller, was a Nazi stormtrooper. Unlike the fanatical Teichmüller, the devotion Witt had toward the Nazi party and their racist pseudoscience is historically disputed and was questioned by Nazi authorities themselves in an assessment [1]. *It is occasionally staggering to remember that the names long immortalized in mathematics belonged to human beings. Even more so to be reminded how small mathematics is within the spectrum of human experience.*

Let $p \geq 2$ be a prime number. We will denote $\mathbb{N} = \{0, 1, 2, \dots\}$. In this section we define a ring scheme over $\operatorname{Spec} \mathbb{Z}$, that is, a commutative ring object in the category of schemes, denoted by $\mathbb{W}$. For a commutative ring R, the functor of points gives a ring $\mathbb{W}(R)$ called the ring of *Witt vectors with coefficients in R.* Some key points before we go in-depth are itemized below:

1. The underlying scheme for $\mathbb{W}$ is $\mathbb{A}^{\mathbb{N}}$, which has a functor of points taking a commutative ring to the set of maps $\mathbb{N} \to R$, written as infinite tuples $(x_0, x_1, x_2, \dots)$ for $x_i \in R$.

2. Let k be a field of characteristic p, and denote $\mathbb{W}_k = \operatorname{Spec} k \times_{\mathbb{Z}} \mathbb{W}$ the base change. Then the relative Frobenius $F = F_{\mathbb{W}_k/k} : \mathbb{W}_k \to \mathbb{W}_k^{(p)}$ and Verschiebung $V = V_{\mathbb{W}_k} : \mathbb{W}_k^{(p)} \to \mathbb{W}_k$ on the underlying abelian group scheme for $\mathbb{W}_k$ are given on points by
$$\begin{aligned} F(A) &: (x_0, x_1, x_2, \dots) \mapsto (x_0^p, x_1^p, x_2^p, \dots) \\ V(A) &: (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots) \end{aligned}$$
for a commutative k-algebra A and elements $x_i \in A$. The relative Frobenius is a homomorphism of ring-schemes over k (this actually always holds), while the Verschiebung is a homomorphism of underlying abelian group schemes

3. If k is a perfect field of characteristic p, we have that $\mathbb{W}(k)$ is a complete DVR with residue field k, and maximal ideal $p\mathbb{W}(k)$. For elements $x_i \in k$, we have the Witt vector $(x_0, x_1, x_2, \dots)$ coincides with the formal sum
$$x_0^\tau + p(x_1^{1/p})^\tau + p^2(x_2^{1/p^2})^\tau + \dots,$$
with all sums taken in the ring $\mathbb{W}(k)$, and $x^\tau = (x, 0, 0, \dots)$ denotes the *Teichmüller representative* of an element $x \in k$.

2.1 The ring of Witt vectors

2.1.1. (Witt polynomials) Define $\Phi_n \in \mathbb{Z}[X_0, X_1, X_2, \dots]$ by
$$\Phi_n = X_0^{p^n} + pX_1^{p^{n-1}} + p^2 X_2^{p^{n-2}} + \dots + p^n X_n,$$
for each $n \in \mathbb{N}$. We regard each Φ_n as a morphism $\mathbb{A}^{\mathbb{N}} \to \mathbb{A}$ of affine schemes with the obvious coordinates for $\mathcal{O}(\mathbb{A}^{\mathbb{N}})$. It is shown in [5, V §1, 1.2] that every morphism $u : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ lifts uniquely to a morphism $\hat{u} : \mathbb{A}^{\mathbb{N}} \times \mathbb{A}^{\mathbb{N}} \to \mathbb{A}^{\mathbb{N}}$ such that
$$\Phi_n \hat{u} = u(\Phi_n \times \Phi_n)$$
for each $n \in \mathbb{N}$. In practice it is not hard to define these lifts inductively.

2.1.2. (The ring of Witt vectors) The affine line $\mathbb{A} = \operatorname{Spec} \mathbb{Z}[x]$ is a commutative ring object in the category of schemes (over $\mathbb{Z}$), a.k.a. a *ring scheme*. As such we have addition and multiplication $\sigma, \pi : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$. In coordinates $\mathbb{A} \times \mathbb{A} = \operatorname{Spec} \mathbb{Z}[X, Y]$, we have σ, π are given by $X + Y$ and XY respectively. By lifting these morphisms, we can define structure maps $\hat{\sigma}, \hat{\pi} : \mathbb{A}^{\mathbb{N}} \times \mathbb{A}^{\mathbb{N}} \to \mathbb{A}^{\mathbb{N}}$ defining $\mathbb{A}^{\mathbb{N}}$ to be a ring scheme. We take the additive and multiplicative identities as $(0, 0, 0, \dots)$ and $(1, 0, 0, \dots)$, realized by maps $\operatorname{Spec} \mathbb{Z} \to \mathbb{A}^{\mathbb{N}}$, respectively.

The uniqueness of lifting in [5, V §1, 1.2] ensures that $\hat{\sigma}, \hat{\pi}$ indeed satisfy the associative, commutative, and distributive properties, as well as identities and inverses, which are necessary to make $\mathbb{A}^n, \hat{\sigma}, \hat{\pi}$ a ring scheme. We denote this ring scheme by $\mathbb{W}$. For any commutative ring R, we also denote $\mathbb{W}(R)$ the ring of R-points, i.e. the ring of morphisms $\mathrm{Mor}(\mathrm{Spec}\, R, \mathbb{W})$. The elements of $\mathbb{W}(R)$ are called *Witt vectors with coefficients in* R, and for Witt vectors $x, y \in \mathbb{W}(R)$, we denote $\hat{\sigma}(x,y) := x \dot{+} y$ and $\hat{\pi}(x,y) = x \dot{\times} y$. Similarly, we take $\dot{-}x$ the additive inverse of x and $x \dot{-} y = x \dot{+} (\dot{-} y)$. Notice, by the fundamental property of our lifts, we have that each $\Phi_n : \mathbb{W} \to \mathbb{A}$ is a homomorphism of ring schemes.

If R is a commutative ring and $a \in R$ is any element, we denote $a^\tau = (a, 0, 0, \dots) \in \mathbb{W}(R)$, called the *Teichmüller representative* of a. For any Witt vector $(a_0, a_1, a_2, \dots) \in \mathbb{W}(R)$ and $a \in R$, one can verify

$$a^\tau \dot{\times} (a_0, a_1, a_2, \dots) = (aa_0, a^p a_1, a^{p^2} a_2, \dots),$$

and hence given $a, b \in R$ we have $(ab)^\tau = a^\tau \dot{\times} b^\tau$, but in general we usually have $(a+b)^\tau \neq a^\tau \dot{+} b^\tau$.

We also define ring schemes $\mathbb{W}_n$, with underlying scheme structure $\mathbb{A}^n$, thought of as the first n coordinates of $\mathbb{W}$, with projection $\mathfrak{R}_n : \mathbb{W} \to \mathbb{W}_n$ a homomorphism of ring schemes. One checks that the lifts $\hat{\sigma}, \hat{\pi}$ indeed give well defined sum and product structures $\mathbb{W}_n \times \mathbb{W}_n \to \mathbb{W}_n$. The points of $\mathbb{W}_n(R)$ are referred to as *Witt vectors of length* n (with coefficients in R).

2.1.3. (Canonical extensions) We have described $\mathbb{W}_n$ as the ring of the first n coordinates of $\mathbb{W}$, and hence we have a canonical projection

$$\mathfrak{R} : \mathbb{W}_{n+1} \to \mathbb{W}_n,$$

a homomorphism of ring schemes for any n. One checks that the canonical projections form an inverse system with $\mathbb{W} \xrightarrow{\sim} \varprojlim_n \mathbb{W}_n$. We also have a canonical embedding

$$\mathfrak{T} : \mathbb{W}_n \to \mathbb{W}_{n+1},$$

a homomorphism of group schemes defined on R-points by

$$\mathfrak{T}(R) : (a_0, \dots, a_{n-1}) \mapsto (0, a_0, \dots, a_{n-1})$$

for Witt vectors with coefficients in a commutative ring R. Passing to the projective limits gives $\mathfrak{T} : \mathbb{W} \to \mathbb{W}$. For any $n, m \in \mathbb{N}$ we have an exact sequence of group schemes

$$0 \to \mathbb{W}_n \xrightarrow{\mathfrak{T}^m} \mathbb{W}_{n+m} \xrightarrow{\mathfrak{R}^m} \mathbb{W}_m \to 0,$$

and similarly in the projective limit we have

$$0 \to \mathbb{W} \xrightarrow{\mathfrak{T}^m} \mathbb{W} \xrightarrow{\mathfrak{R}_m} \mathbb{W}_m \to 0.$$

2.2 Frobenius and Verschiebung revisited

Since $\mathbb{W}$ is merely a ring scheme over $\mathbb{Z}$, we don't have Frobenius morphisms. Instead we denote $\mathfrak{F} : \mathbb{W} \to \mathbb{W}$ to be defined on points by

$$\mathfrak{F}(R) : (a_0, a_1, \dots) \mapsto (a_0^p, a_1^p, \dots)$$

for any commutative ring R and $a = (a_0, a_1, \dots) \in \mathbb{W}(R)$. It should be clear that after base changing to k we have $\mathfrak{F}_k = F_{\mathbb{W}/k} : \mathbb{W}_k \to \mathbb{W}_k$, after noticing that the base change $\mathbb{W}_k^{(p)}$ is isomorphic as a ring scheme over k to $\mathbb{W}_k$, since the underlying scheme is $\mathbb{A}^{\mathbb{N}}$.

Proposition 2.2.1. [5, V, §1, 1.7] Let R be any commutative ring and let $a \in \mathbb{W}(R)$. Now write $b = pa \dot{-} \mathfrak{T}(\mathfrak{F}(a))$. Then we have $b_i - pa_i \in p^2 R$ for each i, where $a = (a_0, a_1, \dots)$ and $b = (b_0, b_1, \dots)$.

Corollary 2.2.2. Let k be a perfect field of characteristic p. Then $\mathbb{W}(k)$ is a complete discrete valuation ring, with residue field k and maximal ideal $p\mathbb{W}(k)$. Every Witt vector $x \in \mathbb{W}(k)$ is uniquely expressible in the form

$$x = x_0^\tau + p x_1^\tau + p^2 x_2^\tau + \dots,$$

for $x_0, x_1, \dots, \in K$.

Notice, by the proposition above it follows by induction that $px = \mathfrak{T}(\mathfrak{F}(x))$ for each $x \in \mathbb{W}(k)$. In particular we have

$$(a_0, a_1, a_2, \dots) = a_0^\tau + p(a_1^{1/p})^\tau + p^2(a_2^{1/p^2})^\tau + \dots,$$

where $a_i^{1/p^i} \in k$ makes sense, as we've assumed k is perfect.

We've identified how the relative Frobenius for Witt vectors $\mathbb{W}_k$ actually descends to schemes over $\mathbb{Z}$ by $\mathfrak{F}$. Our translation map $\mathfrak{T}$ also shows how Verschiebung descends to $\mathbb{Z}$:

Corollary 2.2.3. Let k be any field of characteristic p. The Verschiebung $V_{\mathbb{W}_k} : \mathbb{W}_k \to \mathbb{W}_k$ is identical to the base changed translation $\mathfrak{T}_k : \mathbb{W}_k \to \mathbb{W}_k$. The same holds for the base change of length n vectors $\mathbb{W}_{nk} = \operatorname{Spec} k \times_{\mathbb{Z}} \mathbb{W}_n$.

Proof. It suffices to consider the case $k = \mathbb{F}_p$. In this case we have $\mathfrak{F}_k = F_{\mathbb{W}_k/k}$ is an epimorphism (why?) of affine abelian group schemes over k. Now we have, by the proposition, $\mathfrak{T}_k\mathfrak{F}_k$ is multiplication by p, but also $V_{\mathbb{W}_k}\mathfrak{F}_k$ is multiplication by p. Since $\mathfrak{F}_k$ is epi, we have $V_{\mathbb{W}_k} = \mathfrak{T}_k$. □

Lemma 2.2.4. [5, V, §1, 2.4] Assume k is a perfect field. Let $m, n \in \mathbb{N}$ such that $1 \le n \le m$ and let G an abelian group scheme of finite type, such that the nth iteration $V_G^n : G^{(p)^n} \to G$ vanishes. Then for every homomorphism $f : G \to \mathbb{W}_{mk}$, there exists a unique $g : G \to \mathbb{W}_{nk}$ such that $f = \mathfrak{T}_k^{m-n} g$.

Proof. We have a commutative diagram

$$\begin{tikzcd} G^{(p)^n} \arrow[r, "f^{(p)^n}"] \arrow[d, "V_G^n"'] & \mathbb{W}_{mk}^{(p)^n} \arrow[d, "V_{\mathbb{W}_{mk}}^n"] \\ G \arrow[r, "f"] & \mathbb{W}_{mk} \end{tikzcd}$$

and the vertical arrow on the left vanishes by hypothesis. We have $\mathbb{W}_{mk}^{(p)^n} \cong \mathbb{W}_{mk}$. Under this isomorphism, $V_{\mathbb{W}_{mk}}^n \cong \mathfrak{R}_k^m \circ \mathfrak{T}_k^n$. Then the kernel of $V_{\mathbb{W}_{mk}}^n$ is the isomorphic image of $(\mathbb{W}_{nk})^{(p)^n}$ under $(\mathfrak{T}_k^{m-n})^{(p)^n}$. Now $f^{(p)^n}$ must factor through the kernel, and the absolute Frobenius $F_k : k \to k$ is an isomorphism of rings. It follows that f factors through the image of $\mathbb{W}_{nk}$ under $\mathfrak{T}_k^{m-n}$. ☐

3 Unipotent abelian group schemes

We will denote the category of affine abelian group schemes by $\mathcal{A}$. Then $\mathcal{A}$ forms an abelian category. We denote by $\mathcal{A}^1(A,B)$ the set of isomorphism classes of extensions of objects B by A in the abelian category $\mathcal{A}$. Recall now, we have $\mathcal{A}^1(A,B)$ forms an abelian group under the Baer sum. Under pullbacks and pushforwards of extensions we have $\mathcal{A}^1(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$ is a bifunctor and is bilinear. It follows that $\mathcal{A}^1(A,B)$ has the structure of a $(\mathrm{End}(B), \mathrm{End}(A))$-bimodule.

For the remainder of this paper we will assume that k is a perfect field of characteristic p.

3.1 Witt group extensions

The underlying abelian group scheme on each of $\mathbb{W}_{nk}$ belongs to $\mathcal{A}$. Recall from 2.1.3, we have canonical elements of $\mathcal{A}^1(\mathbb{W}_{mk}, \mathbb{W}_{nk})$ in the form

$$0 \to \mathbb{W}_{nk} \xrightarrow{\mathfrak{T}^m} \mathbb{W}_{(n+m)k} \xrightarrow{\mathfrak{R}^n} \mathbb{W}_{mk} \to 0.$$

Notice $\mathbb{W}_1 = \mathbb{G}_a$, the general additive group scheme over $\mathbb{Z}$. We will abuse notation to say $\mathbb{G}_a$ is also the base change $\operatorname{Spec} k \times_{\mathbb{Z}} \mathbb{G}_a$. For $n \in \mathbb{N}$ we let $e_n \in \mathcal{A}^1(\mathbb{G}_a, \mathbb{W}_{nk})$ denote the canonical extension. We have cocartesian squares

$$\begin{tikzcd} \mathbb{W}_{nk} \arrow[r, "\mathfrak{T}"] \arrow[d, "\mathfrak{T}"'] & \mathbb{W}_{(n+1)k} \arrow[d, "(\mathrm{id}, \mathfrak{R}^n)"] \\ \mathbb{W}_{(n+1)k} \arrow[r, "(\mathrm{id},0)"] & \mathbb{W}_{(n+1)k} \oplus \mathbb{G}_a \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}, \quad \begin{tikzcd} \mathbb{W}_{(n+1)k} \arrow[r, "\mathfrak{T}"] \arrow[d, "\mathfrak{R}"'] & \mathbb{W}_{(n+2)k} \arrow[d, "\mathfrak{R}"] \\ \mathbb{W}_{nk} \arrow[r, "\mathfrak{T}"] & \mathbb{W}_{(n+1)k} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

It follows that $\mathfrak{T} \cdot e_n = 0 \in \mathcal{A}^1(\mathbb{G}_a, \mathbb{W}_{(n+1)k})$ and $\mathfrak{R} e_{n+1} = e_n \in \mathcal{A}^1(\mathbb{G}_a, \mathbb{W}_{nk})$.

Proposition 3.1.1. [5, V, §1, 2.2] Let H be a closed subgroup of $\mathbb{G}_a$ and let $n \geq 1$. Then

a) The pushforward $\mathcal{A}^1(H, \mathbb{W}_{nk}) \to \mathcal{A}^1(H, \mathbb{G}_a)$ of $\mathfrak{R}^{n-1}$ is an isomorphism.

b) The pullback $\mathcal{A}^1(\mathbb{G}_a, \mathbb{W}_{nk}) \to \mathcal{A}^1(H, \mathbb{W}_{nk})$ of $H \hookrightarrow \mathbb{G}_a$ is surjective.

c) $\mathcal{A}^1(\mathbb{G}_a, \mathbb{W}_{nk})$ is free of rank one as a right $\mathrm{End}(\mathbb{G}_a)$-module, with basis element e_n.

Notice in the case $n = 1$, we have $\mathcal{A}^1(G, \mathbb{G}_a)$ is identified with the subgroup of $H^2(G, k)$ given by symmetric cocycles $G \times G \to k$ for any $G \in \mathcal{A}$. The second assertion for $n = 1$ is actually a special case of a more general fact about cohomology of subgroups $H < \mathbb{G}_a$. The rest of the proposition is proven by induction.

Note 3.1.2. If k is not $\mathbb{F}_p$, the ring $\mathrm{End}(\mathbb{G}_a)$ is noncommutative. Generally, it is the skew polynomial ring $k[F]$ where $F\lambda = \lambda^p F$ for each $\lambda \in k$. The element $F \in \mathrm{End}(\mathbb{G}_a)$ is the Frobenius endomorphism, again identifying $\mathbb{G}_a^{(p)} \cong \mathbb{G}_a$.

Proposition 3.1.3. Let G be *unipotent, algebraic*, and abelian. By unipotent we mean that the G is affine and the trivial representation is the unique simple representation, or equivalently that every subgroup scheme $L \leq G$ other than the trivial group has a nonzero homomorphism $L \to \mathbb{G}_a$. By algebraic we mean of finite type. Then there exists $n, r, s, \in \mathbb{N}$ and a left exact sequence

$$0 \to G \to \mathbb{W}_{nk}^r \to \mathbb{W}_{nk}^s.$$

This proposition is proven using ‘artinian induction’ and some techniques using a corollary of the previous proposition on extensions. That is, assuming without loss of generality that G is nontrivial and that for each strict subgroup $L < G$ we have some embedding $L \hookrightarrow \mathbb{W}_{nk}^r$, then a nonzero homomorphism $G \to \mathbb{G}_a$ with nontrivial kernel L gives an extension in $\mathcal{A}^1(H, L)$ for L the kernel of $G \to \mathbb{G}_a$ and $H \leq \mathbb{G}_a$ the image of G. It is argued from b) of the previous proposition that some embedding must then exist for G. By induction this then holds for arbitrary unipotent $G \in \mathcal{A}$. The rest is technical.

3.2 Dieudonné modules

Let $x = (x_0, x_1, x_2, \dots) \in \mathbb{W}(k)$. For $n \in \mathbb{Z}$, denote

$$x^{(p^n)} = (x_0^{p^n}, x_1^{p^n}, x_2^{p^n}, \dots) = \mathfrak{F}^n(x)$$

(negative n makes sense, we have k is perfect). Recall we have $px = \mathfrak{T}(x^{(p)})$ and in particular $p = p \cdot 1 = (0, 1, 0, \dots)$.

Definition 3.2.1. A *Dieudonné module* (over k) is a $\mathbb{W}(k)$-module with the additional structure of two endomorphisms F_M and V_M satisfying the following conditions (denoting $x_M(m) = xm$ for $x \in \mathbb{W}(k)$ and $m \in M$)

$$\begin{cases} F_M x_M = x^{(p)} F_M, \qquad x_M V_M = V_M (x^{(p)})_M, \\ F_M V_M = V_M F_M = p_M. \end{cases}$$

The *Dieudonné ring* $\mathbb{D}_k$ (or simply $\mathbb{D}$) is defined to be the skew-algebra over $\mathbb{W}(k)$ generated by symbols F and V subject to the relations

$$\begin{cases} Fx = x^{(p)}F, \qquad xV = Vx^{(p)}, \qquad x \in \mathbb{W}(k) \\ FV = VF = p. \end{cases}$$

Thus, Dieudonné modules are equivalent to left modules over the ring $\mathbb{D}$.

3.2.2. The ring $\mathbb{D}_k$ is noetherian, has no zero divisors, and is free as a left or right module over $\mathbb{W}(k)$, with basis 1 and positive powers of F and V. For perfect fields K/k, we have also that $\mathbb{D}_K \cong \mathbb{W}(K) \otimes_{\mathbb{W}(k)} \mathbb{D}_k$. Notice the left ideals $\mathbb{D}V$ and $\mathbb{D}F$ are two sided, and $\mathbb{D}/\mathbb{D}V \cong \mathrm{End}(\mathbb{G}_a)$ is the skew polynomial ring over k discussed before, in a natural way identifying the action of F with the relative Frobenius endomorphism. In this way, the action of $V \in \mathbb{D}$ is also identified with the Verschiebung annihilating $\mathbb{G}_a$.

Proposition 3.2.3. For each n, the abelian group scheme $\mathbb{W}_{nk}$ is a functor of $\mathbb{D}$-modules. The homomorphism $\mathbb{D} \to \mathrm{End}(\mathbb{W}_{nk})$ is surjective, and induces and isomorphism

$$\mathbb{D}/\mathbb{D}V^n \to \mathrm{End}(\mathbb{W}_{nk}).$$

Proof. The first claim follows from the $\mathbb{W}(k)$ action on each ring $\mathbb{W}_{nk}(A)$ for commutative k-algebras A, together with Corollary 2.2.3 and Proposition 2.2.1. It is then also clear that V^n vanishes under $\mathbb{D} \to \mathrm{End}(\mathbb{W}_{nk})$ for each n.

Now, for $n = 1$, the isomorphism follows from the previous claim on $\mathrm{End}(\mathbb{G}_a)$. Assume for induction that $\mathbb{D}/\mathbb{D}V^n \to \mathrm{End}(\mathbb{W}_{nk})$ is an isomorphism. Then we have an induced map of short exact sequences

$$\begin{tikzcd}
\mathbb{D}V^n/\mathbb{D}V^{n+1} \arrow[r, hook] \arrow[dd, dashed] & \mathbb{D}/\mathbb{D}V^{n+1} \arrow[r, two heads] \arrow[dd] & \mathbb{D}/\mathbb{D}V^n \arrow[d] \\
 & & \mathrm{End}(\mathbb{W}_{nk}) \arrow[d, dashed, "\mathfrak{T}_*"] \\
\mathrm{Hom}(\mathbb{G}_a, \mathbb{W}_{(n+1)k}) \arrow[r, hook, "\mathfrak{R}^{n*}"] & \mathrm{End}(\mathbb{W}_{(n+1)k}) \arrow[r, two heads, "\mathfrak{T}^*"] & \mathrm{Hom}(\mathbb{W}_{nk}, \mathbb{W}_{(n+1)k})
\end{tikzcd}$$

Indeed, the top row is canonically short exact, and the bottom row is the image of the canonical extension $e_n \in \mathcal{A}^1(\mathbb{G}_a, \mathbb{W}_{nk})$ under $\mathrm{Hom}(-, \mathbb{W}_{(n+1)k})$. Thus we only need surjectivity of $\mathfrak{T}^*$, which follows from a corollary of Proposition 3.1.1. By Lemma 2.2.4, the map $\mathfrak{T}_* : \mathrm{End}(\mathbb{W}_{nk}) \to \mathrm{Hom}(\mathbb{W}_{nk}, \mathbb{W}_{(n+1)k})$ is an isomorphism, and $\mathbb{D}/\mathbb{D}V^n \to \mathrm{End}(\mathbb{W}_{nk})$ is an isomorphism per inductive hypothesis. Then if the induced map $\mathbb{D}V^n/\mathbb{D}V^{n+1} \to \mathrm{Hom}(\mathbb{G}_a, \mathbb{W}_{(n+1)k})$ is an isomorphism, the inductive step follows from the snake lemma. Indeed, by 2.2.4 we also have $\mathfrak{T}_*^n : \mathrm{End}(\mathbb{G}_a) \to \mathrm{Hom}(\mathbb{G}_a, \mathbb{W}_{(n+1)k})$ is an isomorphism. This is essentially the isomorphism $\mathbb{D}/\mathbb{D}V \to \mathbb{D}V^n/\mathbb{D}V^{n+1}$, so we're done. $\square$

3.3 The structure theorem

Consider the directed system

$$\mathbb{W}_{1k} \xrightarrow{\mathfrak{T}} \mathbb{W}_{2k} \xrightarrow{\mathfrak{T}} \mathbb{W}_{3k} \to \cdots.$$

By the proposition 3.2.3, we know each term is a functor of $\mathbb{D}$-modules, and further it is not hard to show that each $\mathfrak{T} : \mathbb{W}_{nk} \to \mathbb{W}_{(n+1)k}$ is a natural homomorphism of $\mathbb{D}$-modules, with $\mathfrak{T}(A)$ isomorphic to $V\mathbb{W}_{(n+1)k}(A) \hookrightarrow \mathbb{W}_{(n+1)k}(A)$, the inclusion of a $\mathbb{D}$-submodule, for each algebra A.

3.3.1. Let U be a unipotent abelian group scheme over k. We denote by $M(U)$ the *Dieudonné module associated to* U, defined by

$$M(U) = \varinjlim_{n} \operatorname{Hom}(U, \mathbb{W}_{nk}).$$

Each U being affine, we have a $\mathbb{D}$-module structure on the set of points $\mathbb{W}_{nk}(U) = \mathbb{W}_{nk}(\mathcal{O}(U))$, and $\operatorname{Hom}(U, \mathbb{W}_{nk})$ is a subset of $\mathbb{W}_{nk}(U) = \operatorname{Mor}(U, \mathbb{W}_{nk})$. One checks, since Verschiebung and Frobenius are both compatible with homomorphisms of abelian group schemes, we indeed have that $\operatorname{Hom}(U, \mathbb{W}_{nk})$ is a $\mathbb{D}$-submodule of $\mathbb{W}_{nk}$. Notice each $\mathfrak{T}$, and hence each $\mathfrak{T}_* = \operatorname{Hom}(U, \mathfrak{T})$, is a monomorphism. We identify $\operatorname{Hom}(U, \mathbb{W}_{nk})$ with its image in $M(U)$. For any homomorphism of unipotent abelian group schemes $f : U \to U'$, we have an induced map $M(f) : M(U') \to M(U)$ of $\mathbb{D}$-modules.

Example 3.3.2. By Lemma 2.2.4, it follows, for each n, that $\operatorname{Hom}(\mathbb{W}_{nk}, \mathfrak{T})$ is an isomorphism of $\mathbb{D}$-modules for $\mathfrak{T} : \mathbb{W}_{mk} \to \mathbb{W}_{(m+1)k}$ whenever $m \geq n$. Therefore $M(\mathbb{W}_{nk}) \cong \operatorname{End}(\mathbb{W}_{nk}) \cong \mathbb{D}/V^n\mathbb{D}$ for each n, by Proposition 3.2.3. Recall the Verschiebung $V_{\mathbb{W}_{nk}}$ factors as $\mathbb{W}_{nk} \xrightarrow{\mathfrak{T}} \mathbb{W}_{(n+1)k} \xrightarrow{\mathfrak{R}} \mathbb{W}_{nk}$. With these identifications, under the functor M this composition becomes

$$\mathbb{D}/\mathbb{D}V^n \to \mathbb{D}/\mathbb{D}V^{n+1} \to \mathbb{D}/\mathbb{D}V,$$

the first map taking the image of 1 in $\mathbb{D}/\mathbb{D}V^n$ to $V \in \mathbb{D}/V^{n+1}$, and the second map being the canonical projection. The composition agrees with multiplication by V on the right for the $\mathbb{D}$-bimodule $M(\mathbb{W}_{nk}) = \mathbb{D}/\mathbb{D}V^n$ (a *left* $\mathbb{D}$-module homomorphism).

A Deiudonné module M is called *effaceable* if it is V-torsion, i.e. if for each $m \in M$ there exists some n such that $V^n m = 0$.

Theorem 3.3.3. The contravariant functor $M(-)$ from unipotent abelian group schemes to (left) $\mathbb{D}$-modules is fully-faithful. The essential image of $M(-)$ is the full subcategory of effaceable $\mathbb{D}$-modules.

Proof.

a) If a unipotent abelian group scheme U is the limit of a inverse system (U_i), then the canonical map $\varinjlim M(U_i) \to M(U)$ is an isomorphism of $\mathbb{D}$-modules.

b) The functor $M(-)$ is left exact, because each $\mathrm{Hom}(-, \mathbb{W}_{nk})$ is left exact. We'll show now that it's exact. Let $\iota : U' \to U$ be a monomorphism of group schemes, for unipotent abelian U', U. We will show $M(\iota) : M(U) \to M(U')$ is surjective. By the above it suffices to assume U is algebraic (and therefore so is U'!), since U is the limit of an inverse system of algebraic unipotent abelian group schemes, and the directed limit of surjections is surjective. Now U is unipotent and algebraic so we may assume that U admits a composition series

$$U' = U_0 \subset U_1 \subset \cdots \subset U_r = U$$

such that every subquotient U_i/U_{i-1} is a subgroup of $\mathbb{G}_a$ for $i > 0$. Let $f \in M(U')$, i.e. $f \in \mathrm{Hom}(U', \mathbb{W}_{nk})$ for some n, identified also with each $\mathfrak{T}^r f \in \mathrm{Hom}(U', \mathbb{W}_{(n+r)k})$. Each $H_i = U_i/U_{i-1}$ being a subgroup of $\mathbb{G}_a$, we can use the theory of extensions in $\mathcal{A}^1(H_i, \mathbb{W}_{mk})$ from Proposition 3.1.1 to conclude by induction that there exists some $g \in \mathrm{Hom}(U_r, \mathbb{W}_{(n+r)k})$ with $M(\iota)(g) = g\iota = \mathfrak{T}^r f$.

c) Here we'll show $M(-)$ is faithfully flat. Let U, H be unipotent abelian group schemes over k. We must show the canonical

$$\phi_H : \mathrm{Hom}(U, H) \to \mathrm{Hom}_{\mathbb{D}}(M(H), M(U))$$

taking each map to the induced map of Diedonné modules is a bijection. In the case $H = \mathbb{W}_{nk}$, this follows after noticing that $\mathrm{Hom}(U, \mathbb{W}_{nk})$ consists precisely of those elements in

$$M(U) = \varinjlim_{m} \mathrm{Hom}(U, \mathbb{W}_{mk})$$

which are annihilated by the left action of V^n. Thus, $\mathrm{Hom}(U, \mathbb{W}_{nk}) \cong \mathrm{Hom}_{\mathbb{D}}(\mathbb{D}/\mathbb{D}V^n, M(U))$. By Proposition 3.1.3, we may 'resolve' any algebraic H by an exact sequence in the form

$$0 \to H \to \mathbb{W}^r_{nk} \to \mathbb{W}^s_{nk}.$$

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(U, \mathbb{W}^r_{nk}) \arrow[r] \arrow[d, "\phi_{\mathbb{W}^r_{nk}}"] & \mathrm{Hom}(U, \mathbb{W}^s_{nk}) \arrow[d, "\phi_{\mathbb{W}^s_{nk}}"] \\
\mathrm{Hom}_{\mathbb{D}}(M(\mathbb{W}^r_{nk}), M(U)) \arrow[r] & \mathrm{Hom}_{\mathbb{D}}(M(\mathbb{W}^s_{nk}), M(U))
\end{tikzcd},$$

with both vertical arrows bijective. Therefore the induced map of kernels, i.e. ϕ_H, is bijective. Passing to inverse limits shows now that ϕ_H is bijective in general.

d) The last step is to determine the essential image. It is clear that each $M(U)$ is indeed effaceable now that we have $M(-)$ is fully faithful, as the V-torsion submodule of $M(U)$ can be calculated as

$$\varinjlim \mathrm{Hom}_{\mathbb{D}}(\mathbb{D}/\mathbb{D}V^n, M(U)) = \varinjlim \mathrm{Hom}(U, \mathbb{W}_{nk}) = M(U).$$

Now let M be an effaceable $\mathbb{D}$-module and assume M is finitely generated, say by r elements. Then M is annihilated by some V^n, and is thus covered as $(\mathbb{D}/\mathbb{D}V^n)^r \twoheadrightarrow M$. Recall $\mathbb{D}$ is noetherian, and hence we can go further and produce some 'resolution' of M as an exact sequence

$$(\mathbb{D}/\mathbb{D}V^n)^s \to (\mathbb{D}/\mathbb{D}V^n)^r \to M \to 0.$$

But since $M(-)$ is fully faithful, we have $(\mathbb{D}/\mathbb{D}V^n)^s \to (\mathbb{D}/\mathbb{D}V^n)^r$ is the image under $M(-)$ of some $f : \mathbb{W}_{nk}^r \to \mathbb{W}_{nk}^s$. By exactness, we realize M as $M(U)$ for the kernel U of f. Passing to inverse limits, we're done.

□

4 Examples

Assume now that k is an algebraically closed field of characteristic p. We are ready to explore Cartier duality via Dieudonné theory. Since the structure theorem applies to unipotent abelian group schemes, the intersection of the relevant classes of group schemes consists precisely of those finite abelian group schemes G such that both G and $G^\sharp$ are unipotent. A finite abelian group scheme G is unipotent if and only if $G^\sharp$ is connected. Connected finite group schemes are often called *infinitesimal groups*. Thus, we are interested in characterizing *infinitesimal unipotent abelian group schemes* as Dieudonné modules.

4.1 Frobenius kernels and fixed points

We'll arrive at the characterization of infinitesimal unipotent abelian group schemes as Dieudonné modules by first considering a non-example:

Example 4.1.1. For any n the group $\mathbb{Z}/p^n$ is unipotent: indeed, $\mathbb{Z}/p$ embeds into $\mathbb{G}_a$ since it is isomorphic to the additive group functor $A \mapsto \{x \in A \mid x^p = x\}$, so we conclude that every nonzero subgroup of $\mathbb{Z}/p^n$ has a nonzero map to $\mathbb{G}_a$. We claim $M(\mathbb{Z}/p^n) \cong \mathbb{D}/\mathbb{D}(V^n, F-1)$. We do this in steps:

a) First recall how each $\mathbb{W}_{nk}$ is isomorphic to the base change $\mathbb{W}_{nk}^{(p)}$ and therefore the relative Frobenius $F_{\mathbb{W}_{nk}/k}$ is really an endomorphism of group schemes. For any endomorphism $\varphi : G \to G$ of a group scheme, we may write G^φ to mean the subfunctor of φ-fixed points, i.e. $G^\varphi(A) = \{x \in G(A) \mid \varphi(x) = x\}$. It is clear that G^φ is a subgroup functor and, group schemes over k being separated, we have that G^φ is a closed subgroup. Now whenever $G^{(p)} \cong G$, we can define G^F to be a closed subgroup of G, identifying F as the relative Frobenius endomorphism.

b) Now we claim $\mathbb{Z}/p^n \cong \mathbb{W}_{nk}^F$. This is easy to see at the level of points, since for any k-algebra A with $\operatorname{Spec} A$ connected, we have that $\mathbb{W}_{nk}^F(A)$ really is just the group $\mathbb{Z}/p^n$. To see this, consider the A-points of $\mathbb{W}_{nk}^F$ which are in the form

$$(x_0, x_1, \dots, x_{n-1})$$

with $x_i^p = x_i \in A$ for each $i = 0, 1, \ldots, n-1$. Since $\operatorname{Spec} A$ is connected, there are p^n such points. But $\mathbb{W}_{nk}^F(A)$ is clearly a cyclic group, since $(1, 0, \ldots, 0)$ is order p^n, noticing that multiplication by p agrees with the shift $\mathfrak{T}$. Thus we can induce up from connected affine schemes to all schemes and establish an isomorphism of functors $\mathbb{W}_{nk}^F \to \mathbb{Z}/p^n$.

c) Notice, each $M(U)$ is actually a $(\mathbb{D}, \operatorname{End}(U))$ bimodule. There is a homomorphism of rings $\mathbb{Z}[F, V]/(FV = VF = p) \to \operatorname{End}(U)$ whenever U descends to the integers (hence $U^{(p)} \cong U$), identifying F, V with the relative Frobenius and Verschiebung endomorphisms. One checks now that for $M = M(U)$ for such U, we have for each $f \in \mathbb{Z}[F, V]/(FV = VF = p)$ the left $\mathbb{D}$-module Mf agrees with $(f)M$, where (f) is the two sided ideal generated by the image of f in $\mathbb{D}$, and IM denotes the left submodule consisting of sums of elements xm for $x \in I, m \in M$ whenever I is a two-sided ideal.

d) Now we want to argue that whenever U is unipotent abelian and descends to the integers that the $\mathbb{D}$-module $M(U^F)$ is isomorphic to $M/\mathbb{D}(F-1)M$ for $M = M(U)$. By the above, $M/\mathbb{D}(F-1)M$ is the same as $M/M(F-1)$, given that $M(F-1)$ makes sense in this context. We have that U^F can be defined via the equalizer diagram

$$U^F \longrightarrow U \underset{\mathrm{id}}{\overset{F_{U/k}}{\rightrightarrows}} U \ .$$

Since $M(-)$ is an exact antiequivalence we have a coequalizer diagram for $M(U^F)$ given as

$$M \underset{\mathrm{id}}{\overset{M(F_{U/k})}{\rightrightarrows}} M \longrightarrow M(U^F) \ .$$

Identifying $M(F_{U/k})$ as right multiplication by F, we have

$$M(U^F) \cong M/M(F-1) \cong M/\mathbb{D}(F-1)M.$$

e) Now it follows: by quotienting $\mathbb{D}$ by the left ideal $\mathbb{D}(V^n, F-1)$, we could instead quotient the ring $\mathbb{D}/\mathbb{D}V^n$ by the left ideal generated by $(F-1)$. Thus, $M(\mathbb{Z}/p^n) = M(\mathbb{W}_{nk}^F) = \mathbb{D}/\mathbb{D}(V, F-1)$.

By the above example, so long as an effaceable Dieudonné module M does not contain any $\mathbb{D}/\mathbb{D}(V^n, F-1)$ as a direct summand, we have that $M \cong M(U)$ for some connected unipotent abelian group scheme U. If in addition M is a finite length module (over $\mathbb{D}$ or over $\mathbb{W}(k)$), it follows that U is then an infinitesimal unipotent abelian group scheme.

The first claim follows from the general structure of abelian group schemes: any quasi-compact abelian group scheme G decomposes as $G_0 \times \Gamma$, where G_0 is the connected component of the identity, and Γ is a finite abelian group, the group of connected components of G. Since $M(-)$ takes direct products of group schemes to direct sums of $\mathbb{D}$-modules, we're done if we know that the group Υ of connected components for any unipotent abelian U is a p-group. Notice Υ, and hence any subgroup of Υ, is unipotent. But the cyclic group of order n, whenever $n > 1$ is not divisible by p, has no nonvanishing homomorphism into $\mathbb{G}_a$, so we're done. The second claim follows from the characterization of *finite* unipotent abelian group schemes as *finite length* Dieudonné modules [5, Corollaire V, §1, 4.4]. Infinitesimal being synonymous with finite and connected means we are done.

Example 4.1.2. For any n and for any r the group $\mathbb{W}_{nk(r)}$ is unipotent: it is a closed subgroup of $\mathbb{W}_{nk}$. Further, $\mathbb{W}_{nk(r)}$ is connected: this is simple to see given $\mathcal{O}(\mathbb{W}_{nk(r)})$ is a local artinian ring. Hence, being finite, we say $\mathbb{W}_{nk(r)}$ is infinitesimal, unipotent, and abelian. We claim $M(\mathbb{G}_{a(r)}) = \mathbb{D}/\mathbb{D}(V^n, F^r)$. We do this in steps:

a) Let U be a unipotent abelian group scheme, and $M = M(U)$. We have an exact sequence defining the r^{th} Frobenius kernel $U_{(r)}$

$$0 \to U_{(r)} \to U \xrightarrow{F^r_{U/k}} U^{(p)^r}$$

and therefore an exact sequence of $\mathbb{D}$-modules

$$M(U^{(p)^r}) \xrightarrow{M(F^r_{U/k})} M(U) \to M(U_{(r)}) \to 0.$$

b) It is simpler to assume U descends to the integers to then claim that $M(U_{(r)}) \cong M(U)/\mathbb{D}(F^r)M(U)$. In general it still makes sense to claim $M(U_{(r)}) \cong M(U)/(M(U)F^r)$, identifying $M(U)F^r$ with the image of the induced map $F^r = M(F^r_{U/k}) : M(U^{(p)^r}) \to M(U)$. Either generality applies and then proves the claim for $M(\mathbb{W}_{nk(r)})$, given how we know $M(\mathbb{W}_{nk}) \cong \mathbb{D}/\mathbb{D}V^n$.

4.2 Applications to representation theory

Recall Proposition 1.2.6, which states how for finite abelian group schemes, the Verschiebung and relative Frobenius are actually dual in the sense of Cartier duality. Notice, a finite unipotent abelian group scheme U necessarily vanishes under some power of the Verschiebung, since $M(U)$ is finite length and effaceable. Therefore, provided U is connected, we must have that $U^{\#}$, being unipotent, must also vanish under some power of the Verschiebung. We conclude infinitesimal unipotent abelian groups vanish under some power of the relative Frobenius as well. If U is infinitesimal unipotent abelian, we define the *height* to be the smallest ℓ such that $F^{\ell}_{U/k}$ vanishes, and similarly the *width* respective to iterations of V_U. We will find in some cases that the isomorphism class of the Cartier dual of G can be deduced simply by understanding how the Frobenius and Verschiebung have interchanged, in a sense, and in particular the height and width have interchanged explicitly.

Our application of Dieudonné theory to representations is purely through Cartier duality. Recall, if G is an affine scheme and V is a representation, by definition this means we have fixed a homomorphism $G \to \mathrm{GL}(V)$, and consequently V is given the structure of a comodule over the coalgebra $\mathcal{O}(G)$, i.e. we have a structure map

$$V \to \mathcal{O}(G) \otimes V$$

defining a 'coaction' by $\mathcal{O}(G)$. If G is finite, the tensor-hom adjunction gives us a structure map

$$\mathcal{O}(G)^* \otimes V \to V$$

defining V to now be a module over $\mathcal{O}(G)^*$. In practice, it is advantageous to reduce questions in representation theory to module theory whenever possible. The cocommutative Hopf algebra $\mathcal{O}(G)^*$ is usually called the *group algebra* of G and is denoted by kG. If G is a finite (discrete) group, the group algebra $\mathcal{O}(G)^*$ agrees with the usual group algebra over k which is 'generated' by elements of the group G.

For finite abelian G, we note $\mathcal{O}(G^\sharp) = \mathcal{O}(G)^*$ per definition, so we have now that representations of G are the same thing as modules over $\mathcal{O}(G^\sharp)$, which may be thought of as sheaves over the scheme $G^\sharp$. Thus, if G is infinitesimal, unipotent, and abelian, Dieudonné theory can help us reduce representations of G into module theory by simply helping us understand the structure of $G^\sharp$.

Example 4.2.1. Let $n < p$. There is an $(n+1)$ dimensional representation L_n of the algebraic group $\mathbb{G}_a$, which we define by identifying $\mathrm{GL}_{n+1} = \mathrm{GL}(L_n)$ and defining $\mathbb{G}_a \to \mathrm{GL}_{n+1}$ on points by

$$a \mapsto \begin{pmatrix} 1 & a & \frac{a^2}{2} & \cdots & \frac{a^n}{n!} \\ 0 & 1 & a & \cdots & \frac{a^{n-1}}{(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

for $x \in \mathbb{G}_a(A)$ given a commutative k-algebra A. By restricting to the Frobenius fixed points and to the first Frobenius kernel, we get some $n+1$ dimensional representations of $\mathbb{Z}/p$ and of $\mathbb{G}_{a(1)} = \alpha_p$. We claim that each restriction of L_n to either closed subgroup is indecomposable (that is, nontrivial and not the direct sum of any nontrivial subrepresentations), and that every indecomposable representation of $\mathbb{Z}/p$ and of $\mathbb{G}_{a(1)}$ is the restriction of L_n for some $0 \leq n < p$. To show this, we use our previous computations of the Cartier duals of $\mathbb{Z}/p$ and α_p, namely μ_p and α_p. We have isomorphisms of algebras

$$k(\mathbb{Z}/p) \cong \mathcal{O}(\mu_p) \cong k[x]/(x^p - 1) \cong k[y]/y^p,$$

the last isomorphism taking x to $y - 1$. A consequence of the structure theorem for modules over a PID is that there is precisely one indecomposable module of each dimension $0 < n+1 \leq p$ over the ring $k[y]/y^p$. The $n+1$ dimensional indecomposable representation is isomorphic to the quotient $k[y]/y^{n+1}$, and the matrix representing the action of y with respect to the ordered basis $1, y, \ldots, y^n$ is nilpotent lower triangular $(n+1)\times(n+1)$ Jordan block. We identify y with the dual element to t with respect to the basis $1, t, \ldots, t^{p-1}$ for $\mathcal{O}(\mathbb{Z}/p) = k[t]/(t^p - t)$. One then checks directly that the action of y on the restriction of L_n describes precisely the same lower triangular Jordan block. Similarly for modules over the group algebra $k(\alpha_p) \cong \mathcal{O}(\alpha_p) = k[t]/t^p$.

Example 4.2.2. Assume that U is an infinitesimal unipotent abelian group scheme such that $A = \mathcal{O}(U)$ is generated by primitive elements (i.e. elements x with coproduct $\Delta(x) = x\otimes 1 + 1\otimes x$). Then we have an isomorphism $A \cong u(\mathfrak{g})$ for some abelian restricted Lie algebra $\mathfrak{g}$. The relationship between $\mathfrak{g}$ and U is that the Cartier dual $U^\sharp$ is the first Frobenius kernel $G_{(1)}$ of a smooth connected group G with tangent space $\mathfrak{g}$. Hence $U^\sharp$ has height 1 and U has width 1. It is known that if G has Krull dimension d, then $\mathfrak{g}$ is d-dimensional as a vector space, and $\mathcal{O}(G_{(1)})$ is isomorphic as an associative algebra to

$$\frac{k[t_1, \ldots, t_d]}{t_1^p, \ldots, t_d^p}.$$

Hence, representations of U are equivalent to modules over some algebra in the above form. For a specific example, this is how we know the algebra structure of $\mathcal{O}(\alpha_p^\sharp)$, without Dieudonné theory. But again, the coalgebra remains mysterious without more computations.

The above are examples of understanding Cartier duals via direct computations in order to reduce representation theory into module theory. But for a general bicommutative Hopf algebra of large dimension, finding generators and relations for the dual is not always easy. Below we see how Dieudonné can help!

Example 4.2.3. For any $n, r \geq 1$, the finite group scheme $\mathbb{W}_{nk(r)}$ has finite representation type if and only if $r = 1$. If $r > 1$, then $\mathbb{W}_{nk(r)}$ has tame representation type if and only if $p^{nr} = 4$, i.e. $r = p = 2$ and $n = 1$. In any other case $p^{nr} > 4$ and $\mathbb{W}_{nk(r)}$ is of wild representation type. Perhaps the best method to see this is to simply compute the Cartier dual and examine its category of sheaves, i.e. modules over the group algebra for $\mathbb{W}_{nk(r)}$. This can be done with Dieudonné theory. We know that $\mathbb{W}_{nk(r)}$ is of height r and width n and further that $M(\mathbb{W}_{nk(r)}) = \mathbb{D}/\mathbb{D}(V^n, F^r)$ (an indecomposable $\mathbb{D}$-module) by the example last section. Thus we know that $\mathbb{W}^\sharp_{nk(r)}$ is of height n and of width r. It follows that the Dieudonné module $M = M(\mathbb{W}^\sharp_{nk(r)})$, being indecomposable, admits an epimorphism $\mathbb{D}/\mathbb{D}(F^n, V^r) \twoheadrightarrow M$. But $\mathbb{D}/\mathbb{D}(F^n, V^r) \cong M(\mathbb{W}_{rk(n)})$ and $M(-)$ is an antiequivalence. Hence we know there is a monomorphism $\mathbb{W}^\sharp_{nk(r)} \hookrightarrow \mathbb{W}_{rk(n)}$. Both finite group schemes $\mathbb{W}_{nk(r)}, \mathbb{W}_{rk(n)}$ have the same dimension p^{nr} of coordinate algebras, and hence so does $\mathbb{W}^\sharp_{nk(r)}$. By dimension, the monomorphism $\mathbb{W}^\sharp_{nk(r)} \hookrightarrow \mathbb{W}_{rk(n)}$ is an isomorphism. Thus the coordinate algebra for $\mathbb{W}^\sharp_{nk(r)}$, i.e. the group algebra for $\mathbb{W}_{nk(r)}$ determining its representations, is isomorphic to

$$\mathcal{O}(\mathbb{W}_{rk(n)}) = \frac{k[t_0, \ldots, t_{r-1}]}{t_0^{p^n}, \ldots, t_{r-1}^{p^n}}.$$

The coalgebra structure is determined inductively with Witt calculus but is irrelevant now for defining modules. These algebras are also isomorphic as associative algebras to the group algebras for the abelian p-groups $(\mathbb{Z}/p^n)^r$. For such groups, the representation type is well known to agree with our claim: finite if and only if cyclic, tame if and only if $n = 1$ and $p = r = 2$, and wild in any other case.

Example 4.2.4. Consider the local algebra $A = k[t]/t^{p^r}$. We know that A is isomorphic as an associative algebra to $\mathcal{O}(\mu_{p^r})$ and to $\mathcal{O}(\alpha_{p^r})$. Since $\mu_{p^r}^\sharp \cong \mathbb{Z}/p^r$ and

$$\alpha_{p^r}^\sharp \cong \mathbb{G}_{a(r)}^\sharp \cong \mathbb{W}_{1k(r)}^\sharp \cong \mathbb{W}_{rk(1)},$$

we have found two different finite abelian unipotent group schemes $\mathbb{Z}/p^r$ and $\mathbb{W}_{rk(1)}$ sharing a group algebra A, and hence sharing an abelian category of representations. In fact any finite group scheme with a group algebra isomorphic to A is going to be unipotent abelian, as A is commutative, and the Cartier dual will then have underlying scheme $\operatorname{Spec} A$, which is connected. Such finite group schemes correspond precisely to the cocommutative Hopf algebra structures on A, and since they are all unipotent, they are all described equivalently as Dieudonné modules.

Dually, we know there is a coalgebra isomorphism between $\mathcal{O}(\mathbb{Z}/p^r)$ and $\mathcal{O}(\mathbb{W}_{rk(1)})$, but more explicitly we know that they are in a generic form

$$\frac{k[t_0, \dots, t_{r-1}]}{(t_0^p - \lambda_0, \dots, t_{r-1}^p - \lambda_{r-1})}$$

for some polynomials λ_i in the r variables with the degree of any given t_j being at most $p-1$, to ensure the algebra is of dimension p^r. For $\mathbb{W}_{rk(1)}$ we have each $\lambda_i = 0$, and for $\mathbb{Z}/p^r$, we have each $\lambda_i = t_i$. Any specialization of the generic form above which is compatible with the fixed coalgebra defined on the monomial basis is defining a closed finite subgroup scheme of $\mathbb{W}_{rk}$. All finite group schemes with group algebra A are in this form, and so to classify them, it is equivalent to find which quotients of $M(\mathbb{W}_{rk})$ correspond to a unipotent group scheme of the desired form.

Recall $M(\mathbb{W}_{rk}) = \mathbb{D}/\mathbb{D}V^r$. We have $\mathbb{Z}/p^r$ is the subgroup of fixed points $\mathbb{W}_{rk}^F$ while $\mathbb{W}_{rk(1)}$ is the Frobenius kernel by definition. As such, we have

$$M(\mathbb{Z}/p^r) \cong \mathbb{D}/\mathbb{D}(F-1, V^r), \quad M(\mathbb{W}_{rk(1)}) \cong \mathbb{D}/\mathbb{D}(F, V^r).$$

In order for U to be a finite group scheme with group algebra A, we must have $M(U)$ is a finite-length quotient of $\mathbb{D}/\mathbb{D}V^r$. Thus we find the action of $F \in \mathbb{D}$ has a minimal polynomial with coefficients in the noncommutative ring $\mathbb{W}_r(k)\{V\}/(V^r, xV = Vx^p)$. It is not hard to see why the minimal polynomial has to be linear in F, given the generic form. Then we know $M(U)$ is isomorphic as a $\mathbb{D}$-module to $\mathbb{D}/\mathbb{D}(F - V^\ell, V^r)$ for some $0 \le \ell \le r$. We have $\ell = r$ gives $\mathbb{W}_{rk(1)}$ while $\ell = 0$ gives $\mathbb{Z}/p^r$. We have shown, including the obvious two, that there are $r+1$ group schemes having group algebra A up to isomorphism.

Notice again, that all $r + 1$ group schemes, by sharing a group algebra, share an abelian category of representations. But by Tannakian duality [4], group schemes are reconstructed from their category of representations, when formalized as *symmetric tensor categories*. Thus we have a profound application of Dieudonné theory to representation theory: an example of classifying all tensor category structures on a given abelian category. Paradoxically, all $r + 1$ tensor categories defined this way have the same ring of representations under the respective tensor product for the corresponding group scheme (exercise!). With this, in a sense, the tensor category is only changing in the symmetric braiding structure. This is far from a general fact about changing the tensor category structure on a fixed abelian category. The ring of representations can change spectacularly; see the author's preprints [2], [3].

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