1 Representations of quivers

1.1 Definitions

Let Q be a quiver, k a field. A **representation** X of Q is given by:

- for each $i \in Q_0$, a k-vector space X_i, and
- for each $i \xrightarrow{\alpha} j \in Q_1$, a linear map $X_\alpha : X_i \to X_j$.

A **morphism** $\theta : X \to Y$ between representations X, Y is given by linear maps $\theta_i : X_i \to Y_i$ for each $i \in Q_0$ such that for all $\alpha \in Q_1$,

$$\begin{array}{ccc} X_i & \xrightarrow{X_\alpha} & X_\alpha \\ \downarrow{\scriptstyle \theta_i} & & \downarrow{\scriptstyle \theta_j} \\ Y_i & \xrightarrow[Y_\alpha]{} & Y_j \end{array}$$

commutes.

Denote by $\mathrm{Hom}(X, Y)$ the vector space of morphisms $X \to Y$.

1.2 Examples

Example 1.1. Let $Q = 1 \longleftarrow 2 \longrightarrow 3$, and consider the representations

$$X = k \xleftarrow{1} k \xrightarrow{1} k \,,$$

$$Y = k \xleftarrow{1} k \longrightarrow 0 \,.$$

What is $\mathrm{Hom}(X, Y)$? Pick $(\theta_1, \theta_2, \theta_3) \in \mathrm{Hom}(X, Y)$. Then the following diagram commutes:

$$\begin{array}{ccccc} k & \xleftarrow{1} & k & \xrightarrow{1} & k \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_3} \\ k & \xleftarrow{1} & k & \longrightarrow & 0 \end{array}$$

Then $\theta_3 = 0$, and for $\theta_1 = \lambda \in k$, $\theta_2 = \lambda$. Thus $\mathrm{Hom}(X, Y) \cong k$.

What is $\mathrm{Hom}(Y, X)$? Pick $(\theta_1, \theta_2, \theta_3) \in \mathrm{Hom}(Y, X)$. Then the following diagram commutes:

$$\begin{array}{ccccc} k & \xleftarrow{1} & k & \xrightarrow{1} & 0 \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_3} \\ k & \xleftarrow{1} & k & \xrightarrow{1} & k \end{array}$$

Then $\theta_3 = 0$,so $\theta_2 = 0$, so $\theta_1 = 0$. Thus $\mathrm{Hom}(Y, X) = 0$.

1.3 Equivalence of categories

Representations of Q with morphisms form a category $\mathrm{Rep}(Q)$. For an algebra A, let $\mathrm{Mod}(A)$ denote the category of left A-modules.

Lemma 1.2. *There is an equivalence of categories*

$$\mathrm{Rep}(Q) \equiv \mathrm{Mod}(kQ).$$

Proof. I will only give the construction. For $M \in \mathrm{Mod}(kQ)$, construct $X \in \mathrm{Rep}(Q)$ as follows:

- $X_i := e_i.M$,
- for $i \xrightarrow{\alpha} j$, $X_\alpha := \alpha.- : e_i.M \to e_j.M$. Note that

$$\alpha.(e_i.m) = (\alpha e_i).m = (e_j\alpha).m = e_j.(\alpha.m) \in e_j.M.$$

For $X \in \mathrm{Rep}(Q)$, construct M as follows:

- $M := \bigoplus_{i \in Q_0} X_i$ as a k-vector space.
- The action of kQ on M is given as follows: for each $i \in Q_0$, let $\iota_i : X_i \hookrightarrow M$ and $\pi_i : M \to X_i$ denote the inclusion and projection maps, respectively. Pick $p = \alpha_n \dots \alpha_3\alpha_2\alpha_1$ a path from i to j. Then for $m \in M$, define

$$p.m = \iota_j \circ X_{\alpha_m} \circ \dots \circ X_{\alpha_2} \circ X_{\alpha_1} \circ \pi_i(m).$$

  Extend the action linearly to kQ.

It remains to check that these constructions give functors inverse to each other. □

Example 1.3. Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and let X be the representation $X_1 \xrightarrow{X_\alpha} X_2 \xrightarrow{X_\beta} X_3$. The corresponding module is $M = X_1 \oplus X_2 \oplus X_3$. How does $\alpha \in kQ$ act on $(x_1, x_2, x_3) \in M$? By definition,

$$\alpha.(x_1, x_2, x_3) = \iota_2 \circ X_\alpha \circ \pi_1(x_1, x_2, x_3) = \iota_2 \circ X_\alpha(x_1) = (0, X_\alpha(x_1), 0).$$

Similarly,

$$\beta\alpha.(x_1, x_2, x_3) = (0, 0, X_\beta X_\alpha(x_1)).$$

It is suggestive to write

$$kQ \cong \begin{pmatrix} k & 0 & 0 \\ kX_\alpha & k & 0 \\ kX_\beta X_\alpha & kX_\beta & k \end{pmatrix},$$

and the action by

$$\begin{pmatrix} k & 0 & 0 \\ kX_\alpha & k & 0 \\ kX_\beta X_\alpha & kX_\beta & k \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

We now ignore (or blur) the distinction between representations of a quiver Q and left modules over the path algebra kQ.

1.4 The simple modules

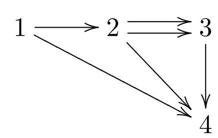
For $i \in Q_0$, consider the representation S_i of Q defined by

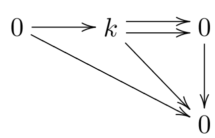
$$(S_i)_j := \delta_{ij} k$$

for all $i \in Q_0$, and

$$(S_i)_\alpha := 0$$

for all $\alpha \in Q_1$.

Example 1.4. Let $Q =$ 

. Then $S_2 =$ 

with all maps zero.

These representations are irreducible (note that the corresponding modules are 1-dimensional). In fact, if there are no cycles in the quiver, these are all the irreducible representations (up to isomorphism).

Lemma 1.5. *If Q has no cycles, then it has $|Q_0|$ irreducible representations (up to isomorphism).*

Proof. Let $|Q_0| = n$. We use the following result from algebra.

Proposition 1.6. *Let R be a ring, I a nilpotent ideal of R (i.e. $I^m = 0$ for $m \gg 0$). If M is a simple left R-module then $IM = 0$.*

Proof. Note that IM is a submodule of M. Since M is simple, $IM = 0$ or $IM = M$. If $IM = M$, then $I^m M = M$ for all m, but $I^m = 0$ for $m \gg 0$: a contradiction. The result follows. □

Thus we have a bijection

$$\{\text{iso. classes of simple modules of } R\} \equiv \{\text{iso. classes of simple modules of } R/I\}.$$

Since Q has no cycles, kQ is finite-dimensional, so $kQ_m = 0$ for $m \gg 0$, so $(kQ_{\geq 1})^m = 0$ for $m \gg 0$: i.e., $kQ_{\geq 1}$ is a nilpotent ideal of kQ. Thus we have a bijection

$$\{\text{iso. classes of simple modules of } kQ\} \equiv \{\text{iso. classes of simple modules of } kQ/kQ_{\geq 1}\}.$$

Note that

$$kQ/kQ_{\geq 1} \cong kQ_0 \cong k^n.$$

Since k^n has n simple modules up to isomorphism, our result follows. □

References

[1] I. Assem, D. Simson, and A. Skowronski, *Elements of the Representation Theory of Associative Algebras: Volume 1.*

[2] W. Crawley-Boevey, *Lectures on Representations of Quivers.*