NAKAYAMA ALGEBRAS

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We follow chapter 5 of [1]. Nakayama algebras are finite-dimensional and representation-finite algebras that have a nice representation theory in the sense that the finite-dimensional indecomposable modules are easy to describe. In particular, we will show that these algebras are characterized by the property that any indecomposable module has a unique composition series. For a basic and connected algebra, Nakayama is equivalent to an easily-checked condition on the underlying quiver.

Throughout these notes, A is a finite-dimensional algebra over a field k and $A-\text{mod}$ is the category of finite-dimensional left A-modules.

1. Loewy length

For $M \in A - \text{mod}$, define the **radical series** of M to be

$$0 \subset \cdots \subset \text{rad}^2 M \subset \text{rad} M \subset M.$$

For $M \neq 0$, $\text{rad}M$ is properly contained in M, and since $\dim_k M < \infty$, the radical series of M is finite. We denote by $r\ell(M)$ the length of the radical series of M. Note that $\text{rad}^i M = (\text{rad}A)^i.M$, so $\text{rad}^i A = (\text{rad}A)^i$ and $r\ell(M) \leq r\ell(A)$.

Define the **socle series** of M inductively: $\text{soc}^0 M := 0$, and

$$\text{soc}^{i+1} M := \pi^{-1}\text{soc}(M/\text{soc}^i M)$$

where $\pi : M \to M/\text{soc}^i M$ is the quotient map, i.e.

$$\text{soc}^{i+1} M/\text{soc}^i M \cong \text{soc}(M/\text{soc}^i M).$$

Since $\dim_k M < \infty$, $\text{soc}M \neq 0$ if $M \neq 0$ and the socle series

$$0 \subset \text{soc}M \subset \text{soc}^2 M \subset \cdots \subset M$$

is finite. Denote by $s\ell(M)$ the length of the socle series of M.

Remark 1.1. For $i \geq 1$, $\text{soc}^{i+1} M$ is the pull-back of $M \xrightarrow{\pi} M/\text{soc}^i M \hookleftarrow \text{soc}(M/\text{soc}^i M)$:

$$(1) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \text{soc}^i M & \longrightarrow & \text{soc}^{i+1} M & \longrightarrow & \text{soc}(M/\text{soc}^i M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{soc}^i M & \longrightarrow & M & \xrightarrow{\pi} & M/\text{soc}^i M & \longrightarrow & 0
\end{array}$$

Lemma 1.2. *Let $M \in A - \text{mod}$. For $m \in M$ and $i \geq 1$, $m \in \text{soc}^i M$ if and only if* $\text{rad}^i A.m = 0$.

Proof. We use induction. Suppose the result holds for all $i \leq n$. For $\pi : M \to M/\text{soc}^n(M)$ the quotient map,

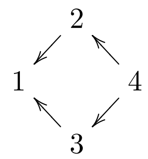
$$\begin{array}{ccc} m \in \text{soc}^{n+1} M \Longleftrightarrow \pi(m) \in \text{soc}(M/\text{soc}^n(M)) & \Longleftrightarrow & \text{rad}A.\pi(m) = 0 \\ & & \Updownarrow \\ \text{rad}^{n+1} A.m = 0 & \Longleftrightarrow & \text{rad}A.m \subset \text{soc}^n M \end{array}$$

Thus the result holds for $i = n + 1$. It remains to show that $m \in \text{soc}M \Leftrightarrow \text{rad}A.m = 0$.

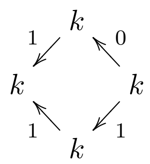
Suppose $m \in \text{soc}M$. Then $m \in \sum_j S_j$ a finite sum of nonzero simple submodules of M. By Nakayama's lemma, $\text{rad}A.S_j \neq S_j$, so $\text{rad}A.S_j = 0$ for each j. Thus $\text{rad}A.m = 0$.

Suppose $\text{rad}A.m = 0$. Let $N = A.m$ the cyclic submodule generated by m. Note that $\text{rad}N = \text{rad}A.(A.m) = 0$, so $N \cong N/\text{rad}N$ is semisimple. Thus $N \subset \text{soc}M$, i.e. $m \in \text{soc}M$. $\square$

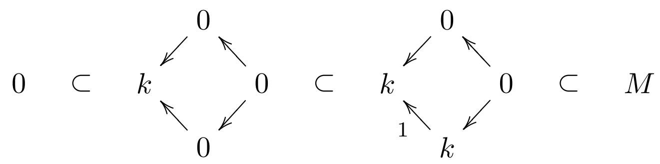
Example 1.3. Let Q be the quiver



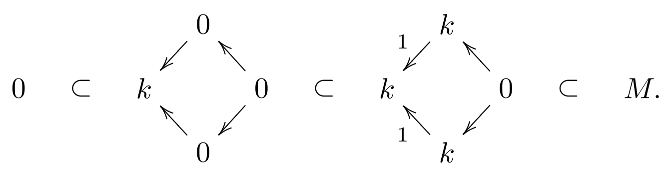
and $A = kQ$ the path algebra. Let M be the representation



Then M has radical series



and socle series



Note that the series are different. However, it is true that $s\ell(M) = r\ell(M)$ in general, which we now show.

Lemma 1.4 (V.1.1)**.** *If $f : M \to N$ is a morphism in $A - \text{mod}$, then $f(\text{rad}^i M) \subset \text{rad}^i N$ for all $i \geq 0$. If f is epic, then $f(\text{rad}^i M) = \text{rad}^i N$ for all $i \geq 0$.*

Proof. We use induction. The result holds for $i = 0$. Suppose the result for i. Then

$$f(\text{rad}^{i+1} M) = f\left(\text{rad}(\text{rad}^i M)\right) = f(\text{rad}A.\text{rad}^i M) = \text{rad}A.f(\text{rad}^i M).$$

The result follows since $\text{rad}A.N = \text{rad}N$. □

Corollary 1.5 (V.1.2)**.** *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence in $A - \text{mod}$. Then $r\ell(M) \geq \max\{r\ell(L), r\ell(N)\}$.*

Proof. By the previous result, $f(\text{rad}^i L) \subseteq \text{rad}^i M$ and $g(\text{rad}^i M) = \text{rad}^i N$. So $\text{rad}^i M = 0$ implies $\text{rad}^i L = \text{rad}^i N = 0$. □

Remark 1.6. In the previous result, exactness at M is not required.

Recall the duality functor $D : A-\text{mod} \to A^{op}-\text{mod}$, $DM = \text{Hom}_k(M, k)$.

Lemma 1.7. *For $M \in A - \text{mod}$ and $i \geq 0$, $\text{soc}^i DM \cong D(M/\text{rad}^i M)$.*

Proof. Since $\text{soc}^0 DM = 0$ and $\text{rad}^0 M = M$, the result holds for $i = 0$. Now suppose $i \geq 1$. Note that

$$D(M/\text{rad}^i M) \cong \ker(DM \to D\text{rad}^i M, f \mapsto f\iota)$$

where ι is the inclusion $\text{rad}^i M \hookrightarrow M$.

Suppose $f \in DM$ such that $f\iota = 0$. For $a \in \text{rad}^i A$ and $m \in M$,

$$f.a(m) = f(a.m) = f\iota(a.m) = 0.$$

Thus $f.\text{rad}^i A = 0$, so by Lemma 1.2, $f \in \text{soc}^i DM$.

Suppose $f \in \text{soc}^i DM$. Then by Lemma 1.2, $f.(\text{rad}A)^i = 0$. For $a.m \in (\text{rad}A)^i.M = \text{rad}^i M$,

$$f\iota(a.m) = f\iota.a(m) = (f.a)\iota(m) = 0,$$

so $f\iota = 0$. The result follows. □

Corollary 1.8. *For $M \in A - \text{mod}$, $s\ell(DM) = r\ell(M)$.*

Proof. By the previous result, $\text{soc}^n DM = DM$ if and only if $M/\text{rad}^n M = M$, that is, $\text{rad}^n M = 0$. □

Proposition 1.9 (V.1.3)**.** *For $M \in A - \text{mod}$, $r\ell(M) = s\ell(M)$.*

Proof. We first prove that $s\ell(M) \leq r\ell(M)$ by induction on $s\ell(M)$. Since

$$s\ell(M) = 0 \Leftrightarrow M = 0 \Leftrightarrow r\ell(M) = 0,$$

the result holds for $s\ell(M) = 0$.

Suppose $s\ell(X) \leq r\ell(X)$ for all $X \in A-\text{mod}$ such that $s\ell(X) = i \geq 0$ and suppose $s\ell(M) = i+1$. Then $r\ell(M) = j > 0$ and $\text{rad}^{j-1}M$ is semisimple since $\text{rad}\,\text{rad}^{j-1}M = 0$. Thus $\text{rad}^{j-1}M \subset \text{soc}M$, so there is an epimorphism

$$M/\text{rad}^{j-1}M \twoheadrightarrow M/\text{soc}M.$$

By Lemma 1.4,

$$r\ell(M/\text{rad}^{j-1}M) \geq r\ell(M/\text{soc}M).$$

By the induction hypothesis, $s\ell(M/\text{soc}M) \leq r\ell(M/\text{soc}M)$. Since

$$\text{rad}(M/\text{rad}^{j-1}M) \cong \text{rad}M/\text{rad}^{j-1}M,$$

we have that

$$r\ell(M/\text{rad}^{j-1}M) = r\ell(M) - 1,$$

and since $\text{soc}(M/\text{soc}M) \cong \text{soc}^2M/\text{soc}M$,

$$s\ell(M/\text{soc}M) = s\ell(M) - 1.$$

Then

$$r\ell(M) - 1 \geq r\ell(M/\text{soc}M) \geq s\ell(M/\text{soc}M) = s\ell(M) - 1.$$

Thus $s\ell(M) \leq r\ell(M)$.

By Corollary 1.8,

$$r\ell(M) = s\ell(DM) \leq r\ell(DM) = s\ell(DDM) = s\ell(M).$$

Thus $r\ell(M) = s\ell(M)$. □

Definition 1.10. We define the **Loewy length** $\ell\ell(M) := r\ell(M) = s\ell(M)$.

Since $\text{rad}(M \oplus N) = \text{rad}M \oplus \text{rad}N$, we have that $\ell\ell(M_1 \oplus \cdots \oplus M_n) = \max\{\ell\ell(M_1), \ldots, \ell\ell(M_n)\}$.

2. Uniserial modules and algebras

Definition 2.1. We say $M \in A - \text{mod}$ is **uniserial** if it has a unique composition series, i.e. if the submodule lattice of M is a chain.

If M is uniserial, then so is any submodule and any quotient of M, and M is indecomposable.

Remark 2.2. If $M \in A - \text{mod}$ is uniserial, then M has a unique maximal submodule, namely $\text{rad}M$, and a unique simple submodule, namely $\text{soc}M$.

Remark 2.3. The book now says that a uniserial module is determined by its composition series up to isomorphism, that is, if M and N are uniserial modules that have the same composition factors in the same place, then $M \cong N$. The book goes on to say that the proof is an obvious induction, but I don't see it.

Lemma 2.4 (V.2.2)**.** *Suppose $M \in A - \text{mod}$. The following are equivalent:*

(1) M is uniserial,
(2) the radical series of M is a composition series,
(3) the socle series of M is a composition series,

(4) $\ell(M) = \ell\ell(M)$.

Proof. ($1 \Rightarrow 3$) Suppose M is uniserial. Since $M/\mathrm{soc}^i M$ is uniserial,

$$\mathrm{soc}^{i+1}M/\mathrm{soc}^i M \cong \mathrm{soc}(M/\mathrm{soc}^i M)$$

is simple.

($3 \Rightarrow 4$) clear.

($4 \Rightarrow 2$) Let $n = \ell(M) = \ell\ell(M)$. If $n = 0$ or 1, the radical series is a composition series, so suppose $n > 1$. Consider the exact sequence

$$0 \to \mathrm{rad}M \to M \to M/\mathrm{rad}M \to 0.$$

Then $\ell(M) = \ell(M/\mathrm{rad}M) + \ell(\mathrm{rad}M)$. Continuing in this fashion, we get

$$\ell(M) = \sum_{i=0}^{n-1} \ell(\mathrm{rad}^i M/\mathrm{rad}^{i+1}M) = n.$$

For $0 \leq i < n-1$, $\mathrm{rad}^i M$ is nonzero so $\mathrm{rad}^i M/\mathrm{rad}^{i+1}M$ is nonzero. Then $\ell(\mathrm{rad}^i M/\mathrm{rad}^{i+1}M)$.

($2 \Rightarrow 1$) Suppose the radical series

$$0 = \mathrm{rad}^n M \subset \cdots \mathrm{rad}^2 M \subset \mathrm{rad}M \subset M$$

is a composition series, and let

$$0 = N_n \subset \cdots N_2 \subset N_1 \subset M$$

be a composition series. We show by induction that $N_i = \mathrm{rad}^i M$ for all $0 \leq i \leq n$. The result holds for $i = 0$. Suppose the result holds for some $0 \leq i < n$. Since the radical series is a composition series, $\mathrm{rad}^i M/\mathrm{rad}^{i+1}M$ is simple, so $N_i = \mathrm{rad}^i M$ has a unique maximal submodule, namely $\mathrm{rad}^{i+1}M$. Thus $N_{i+1} = \mathrm{rad}^{i+1}M$, and M is uniserial. □

Definition 2.5. We say A is **left (resp. right) serial** if every indecomposable projective left (resp. right) A-module is uniserial.

Lemma 2.6 (V.2.5)**.** *An algebra A is left serial if and only if for each indecomposable projective P, $\mathrm{rad}P/\mathrm{rad}^2 P$ is simple or zero.*

Proof. ($\Rightarrow$) By Lemma 2.4, the radical series of P is a composition series.

($\Leftarrow$) Consider the radical series

$$0 = \mathrm{rad}^n P \subset \cdots \subset \mathrm{rad}^2 P \subset \mathrm{rad}P \subset P.$$

We show by induction that $\mathrm{rad}^{i-1}P/\mathrm{rad}^i P$ is simple or zero for $1 \leq i < n$. The result holds for $i = 1$ by (I.5.17) and for $i = 2$ by hypothesis.

Suppose the result holds for some $2 \leq i < n$. Let $f : P' \to \mathrm{rad}^{i-1}P$ be a projective cover and $\pi : \mathrm{rad}^{i-1}P \to \mathrm{rad}^{i-1}P/\mathrm{rad}^i P$ the quotient map. Note that πf is surjective and $\ker \pi f = f^{-1}\mathrm{rad}^i P$. By Lemma 1.4, $f(\mathrm{rad}P') = \mathrm{rad}^i P$, and if $f(p_1) = f(p_2) \in \mathrm{rad}^i P$ for $p_1 \in \mathrm{rad}P'$, then $p_1 - p_2 \in \ker f$. Thus $\ker \pi f = \mathrm{rad}P' + \ker f$, so $\ker \pi f$ is minimal and $\pi f : P' \to \mathrm{rad}^{i-1}P/\mathrm{rad}^i P$ is a projective cover. By the induction hypothesis, $\mathrm{rad}^{i-1}P/\mathrm{rad}^i P$ is simple so P' is indecomposable by (I.5.17). From Lemma

1.4 we get epimorphisms $f_1 : \mathrm{rad}P' \to \mathrm{rad}^i P$ and $f_2 : \mathrm{rad}^2 P' \to \mathrm{rad}^{i+1} P$ by restricting f. There is an epimorphism $h : \mathrm{rad}P'/\mathrm{rad}^2 P' \to \mathrm{rad}^i P/\mathrm{rad}^{i+1} P$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{rad}^2 P' & \longrightarrow & \mathrm{rad}P' & \longrightarrow & \mathrm{rad}P'/\mathrm{rad}^2 P' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \mathrm{rad}^{i+1} P & \longrightarrow & \mathrm{rad}^i P & \longrightarrow & \mathrm{rad}^i P/\mathrm{rad}^{i+1} P & \longrightarrow & 0 \end{array}$$

commute. Since P' is indecomposable projective, $\mathrm{rad}P'/\mathrm{rad}^2 P'$ is simple or zero by the induction hypothesis. Thus so is $\mathrm{rad}^i P/\mathrm{rad}^{i+1} P$. □

Theorem 2.7 (V.2.6)**.** *A basic k-algebra A is left serial if and only if for every vertex a in the underlying quiver Q_A of A, there is at most one arrow with source a.*

Proof. By Lemma 2.6, A is left serial if and only if, for every $a \in (Q_A)_0$, the left A-module

$$\mathrm{rad}P(a)/\mathrm{rad}^2 P(a) \cong (\mathrm{rad}A/\mathrm{rad}^2 A)e_a$$

is simple or zero, i.e. 1-dimensional since A is basic. The result follows since

$$(\mathrm{rad}A/\mathrm{rad}^2 A)e_a \cong \bigoplus_{b \in (Q_A)_0} e_b(\mathrm{rad}A/\mathrm{rad}^2 A)e_a$$

and

$$\dim_k e_b(\mathrm{rad}A/\mathrm{rad}^2 A)e_a = |\{a \to b \in (Q_A)_1\}|.$$

□

Corollary 2.8. *A basic k-algebra A is right serial if and only if for every vertex a in the underlying quiver Q_A of A, there is at most one arrow with sink a.*

Proof. Since the right projective A-modules are the left projective A^{op}-modules, A is right serial if and only if A^{op} is left serial. The result follows from the theorem since $Q_{A^{op}} = (Q_A)^{op}$. □

Remark 2.9. The results above give conditions only on the underlying quiver, not on the admissible ideals (except that the algebra need be finite-dimensional).

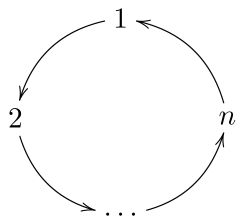
3. Nakayama algebras

Definition 3.1. We say A is a **Nakayama algebra** if it is both left and right serial, i.e. the indecomposable projectives and indecomposable injectives are uniserial.

Theorem 3.2 (V.3.2)**.** *A basic and connected algebra A is a Nakayama algebra if and only if the underlying quiver Q_A is*

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

or



Proof. This follows from Theorem 2.7 and Corollary 2.8. □

Remark 3.3. This previous result is a condition simply on the underlying quiver of A (except that for the second quiver, a power of the cycle has to be in the admissible ideal since A is finite-dimensional).

Lemma 3.4 (V.3.3)**.** *Let A be an algebra and J a proper 2-sided ideal.*

(1) If A is left (or right) serial then so is A/J.
(2) If A is Nakayama then so is A/J.

Proof. Suppose A is left serial. Write $A = \bigoplus_i P_i$, each P_i indecomposable. Then $A/J \cong \bigoplus_i P_i/JP_i$. Since A is left serial, P_i is uniserial, thus so is P_i/JP_i. Then P_i/JP_i is indecomposable, so A/J is left serial.

The result for right serial follows similarly, and 2 follows easily from 1. □

Note that $\mathrm{soc}M \cong \mathrm{soc}E(M)$ for E the injective envelope of M.

Lemma 3.5 (V.3.4)**.** *Let A be Nakayama and $P \in A-\mathrm{mod}$ an indecomposable projective such that $\ell\ell(P) = \ell\ell(A)$. Then P is also injective.*

Proof. Let $u : P \to E$ be an injective envelope. Since P is uniserial, $\mathrm{soc}P$ is simple, thus so is $\mathrm{soc}E \cong \mathrm{soc}P$. Thus E is indecomposable. Since A is Nakayama, E is uniserial and

$$\ell\ell(A) = \ell(P) \leq \ell(E) = \ell\ell(E) \leq \ell\ell(A).$$

Thus $\ell(P) = \ell(E)$ and $P \cong E$. □

Theorem 3.6 (V.3.5)**.** *Let A be Nakayama, $M \in A-\mathrm{mod}$ indecomposable and $t = \ell\ell(M)$. There exists an indecomposable projective $P \in A-\mathrm{mod}$ such that $M \cong P/\mathrm{rad}^t P$. In particular, A is representation-finite.*

Remark 3.7. The book supposes in addition that A is basic and connected. I don't see where these conditions are used.

Proof. Since $\ell\ell(M) = t$, $\mathrm{rad}^t M = \mathrm{rad}^t A.M = 0$ so M is naturally a left $A/\mathrm{rad}^t A$-module (write $B = A/\mathrm{rad}^t A$). Since $\mathrm{rad}^{t-1} \neq 0$, $\mathrm{rad}^{t-1} A \neq 0$ so $\ell\ell(B) = t$. Since A is Nakayama, B is Nakayama by Lemma 3.4, and we decompose B into its indecomposable projectives

$$B \cong \oplus_i P_i/\mathrm{rad}^t P_i$$

where $A = \oplus_i P_i$ with each P_i indecomposable. Let $f : \oplus_{j=1}^r P'_j \to M$ be a projective cover in $B-$mod with each P'_j indecomposable. Since

$$t = \ell\ell(B) \geq \max\{\ell\ell(P'_1), \ldots, \ell\ell(P'_r)\} \geq \ell\ell(M) = t,$$

$\ell\ell(P'_j) = t$ for some j. Rearrange the P'_js so that $\ell\ell(P'_j) = t$ for all $j \leq s$, so $\ell\ell(P'_j) < t$ for all $j > s$.

Write $f_j = f|_{P'_j}$. Suppose no f_j is injective for $j \leq s$. Then

$$\ell\ell(\mathrm{Im} f_j) = \ell\ell(P'_j/\mathrm{Ker} f_j) < t$$

for all j. Since

$$\bigoplus_{j=1}^{r} \mathrm{Im} f_j \to M$$

is surjective, $\ell\ell(M) < t$ by Lemma 1.4, a contradiction. Thus f_q is injective for some $q \leq s$. By Lemma 3.5, P'_j is injective since $\ell\ell(P'_q) = t = \ell\ell(B)$. Thus f_q is a section. Since M is indecomposable, f_q is an isomorphism, and

$$M \cong P'_q = P_i/\mathrm{rad}^t P_i$$

for some i. □

Corollary 3.8. *An algebra A is Nakayama if and only if every indecomposable A-module is uniserial.*

Example 3.9. Let Q be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

with relation $\gamma\beta\alpha = 0$. Then *all* the indecomposable A-modules are

i	P_i	$P_i/\mathrm{rad}P_i$	$P_i/\mathrm{rad}^2 P_i$	$P_i/\mathrm{rad}^3 P_i$
1	$kkk0$	$k000$	$kk00$	P_1
2	$0kkk$	$0k00$	$0kk0$	P_2
3	$00kk$	$00k0$	P_3	
4	$000k$	P_4		

References

[1] Ibrahim Assem, Daniel Simon, Andrzel Skowronski, *Elements of the representation theory of associative algebras.*