QUIVERS AND PATH ALGEBRAS

JIM STARK

1. Definitions

Definition 1. A quiver Q is a finite directed graph. Specifically $Q = (Q_0, Q_1, s, t)$ consists of the following four data:

- A finite set Q_0 called the *vertex set*.
- A finite set Q_1 called the *edge set*.
- A function $s\colon Q_1 \to Q_0$ called the *source function*.
- A function $t\colon Q_1 \to Q_0$ called the *target function*.

This is nothing more than a finite directed graph. We allow loops and multiple edges. The only difference is that instead of defining the edges as ordered pairs of vertices we define them as their own set and use the functions s and t to determine the source and target of an edge.

Definition 2. A (possibly empty) sequence of edges $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ is called a *path* in Q if $t(\alpha_i) = s(\alpha_{i+1})$ for all appropriate i. If p is a non-empty path we say that the *length* of p is $\ell(p) = n$, the source of p is $s(\alpha_1)$, and the target of p is $t(\alpha_n)$. For an empty path we must choose a vertex from Q_0 to be both the source and target of p and we say $\ell(p) = 0$.

Note that paths are read right to left as in composition of functions. Though the source and target functions of a quiver are defined on edges and not on paths we will abuse notation and write $s(p)$ and $t(p)$ for the source and target of a path p. If p and q are paths in Q such that $t(q) = s(p)$ then we can form the composite path pq. This path is defined by appending the possibly empty sequence of edges in p to the end (left) of the possibly empty sequence of edges in q giving a new path of length $\ell(pq) = \ell(p) + \ell(q)$. This operation is clearly associative.

An empty path whose source and target are the vertex $i \in Q_0$ is called the trivial path at i and is denoted e_i. Note that the composition of paths $e_i e_i$ is length zero starting at i therefore $e_i^2 = e_i$. Also note that if $i \neq j$ then e_i and e_j cannot be composed as paths. In the definition to come this implies that $e_i e_j = 0$.

Definition 3. Let k be a field and Q a quiver. Define kQ to be the k-vector space that has as its basis the set of all paths in Q. If p and q are two paths in Q define their product pq to be the composition of the paths p and q if $t(q) = s(p)$ and 0 otherwise. We extend this operation to arbitrary vectors in kQ by distributivity. As composition of paths is associative this gives kQ the structure of an associative k-algebra. It is called the *path algebra* of the quiver Q.

We identify Q_0 and Q_1 with the set of all paths of length 0 and the set of all paths of length 1 respectively. In general we define Q_n to be the set of all paths in

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Q of length n and kQ_n to be the linear subspace of kQ spanned by the Q_n. As a vector space kQ is then the direct sum

$$kQ = \bigoplus_{i \in \mathbb{N}_0} kQ_i.$$

If $p \in Q_n$ and $q \in Q_m$ then either $pq = 0$ or $\ell(pq) = \ell(p) + \ell(q) = n + m$. In either case $pq \in kQ_{n+m}$ therefore $(kQ_n)(kQ_m) \subseteq kQ_{n+m}$. This shows that kQ is a graded k-algebra.

From the decomposition $kQ = \bigoplus_{i \in \mathbb{N}_0} kQ_i$ we also immediately see that kQ is finite dimensional if and only if Q contains no cycles.

2. Examples

Example 1. Let Q be the quiver

$$1$$

with vertex set $\{1\}$ and no edges. The trivial path at 1 is then the only path in Q. For $a, b \in k$ we find $(ae_1)(be_1) = abe_1$ therefore mapping $e_1 \mapsto 1$ gives $kQ = k$.

Example 2. Let Q be the quiver

$$1 \qquad 2 \qquad \cdots \qquad n\ .$$

The only paths are the trivial paths $\{e_i\}_{i=1}^n$. Mapping e_i to the i^{th} standard basis vector of the product ring k^n gives $kQ = k^n$.

Example 3. Let Q be the quiver

$$1 \xrightarrow{\alpha} 2\ .$$

The paths in Q are $\{e_1, e_2, \alpha\}$ which leads to the multiplication shown in Table 1.

Table 1

	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_2	α
α	α	0	0

Define $kQ \to \mathbb{M}_2(k)$ by extending $e_1 \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $e_2 \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and $\alpha \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ linearly. These matrixes satisfy the same multiplication table as $\{e_1, e_2, \alpha\}$ so this is a well defined map of algebras. We map basis elements to basis elements therefore it is a bijection onto its image giving

$$kQ = \{A \in \mathbb{M}_2(k) \mid A_{12} = 0\} = \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}.$$

Example 4. Let Q be the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n\ .$$

Observe that for $i \leq j$ there is a unique path from i to j; let α_{ij} be this path. Then the paths of Q are $\{\alpha_{ij}\}_{i \leq j}$. Let $E_{ij} \in \mathbb{M}_n(k)$ be the matrix whose only non-zero entry is a 1 in the $(ij)^{\text{th}}$ position. Defining $kQ \to \mathbb{M}_n(k)$ by $\alpha_{ij} \mapsto E_{ij}$ and applying the previous argument gives kQ equal to the algebra of lower triangular matrixes contained in $\mathbb{M}_n(k)$.

The previous two examples immediately generalize to the following result. If Q is a quiver with n vertices and the property that for any two vertices i and j there is at most one path from i to j, then the path algebra of Q is

$$kQ = \{A \in \mathbb{M}_n(k) \mid A_{ij} = 0 \text{ if there is no path from } i \text{ to } j\}\,.$$

This condition on Q is very restrictive but for the quivers that satisfy it we can immediately compute kQ.

Example 5. Let Q be the quiver

$$\begin{tikzcd} & 2 \arrow[d] & \\ 1 \arrow[r] & 5 & 3 \arrow[l] \\ & 4 \arrow[u] & \end{tikzcd} ,$$

then

$$kQ = \begin{bmatrix} k & 0 & 0 & 0 & k \\ 0 & k & 0 & 0 & k \\ 0 & 0 & k & 0 & k \\ 0 & 0 & 0 & k & k \\ 0 & 0 & 0 & 0 & k \end{bmatrix}.$$

Example 6. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha"] \end{tikzcd}$$

The paths of Q are $\left\{e_1, \alpha, \alpha^2, \ldots\right\}$. The identity element of kQ is simply e_1 therefore the map $kQ \to k[x]$ defined by $e_1 \mapsto 1$ and $\alpha \mapsto x$ is an isomorphism; $kQ = k[x]$.

Example 7. Let Q be the quiver

$$\begin{tikzcd} \cdots & 1 \arrow[loop above, "\alpha_2"] \arrow[loop right, "\alpha_1"] \arrow[loop below, "\alpha_n"] \arrow[loop left] \end{tikzcd}$$

consisting of a single vertex and n loops. Then mapping $e_1 \mapsto 1$ and $\alpha_i \mapsto x_i$ gives $kQ = k\langle x_1, \ldots, x_n\rangle$, the free algebra on n non-commuting variables.

Example 8. Let Q be the Kronecker quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd} .$$

The paths are $\{e_1, e_2, \alpha, \beta\}$ with multiplication given in Table 2.

Table 2

	e_1	e_2	α	β
e_1	e_1	0	0	0
e_2	0	e_2	α	β
α	α	0	0	0
β	β	0	0	0

Define the *Kronecker Algebra* to be $K = \left[\begin{smallmatrix} k & 0 \\ k^2 & k \end{smallmatrix}\right]$ where we multiply elements from k and k^2 using the standard scalar multiplication of a field on a two dimensional vector space. Explicitly

$$\begin{bmatrix} a & 0 \\ (b,c) & d \end{bmatrix}\begin{bmatrix} e & 0 \\ (f,g) & h \end{bmatrix} = \begin{bmatrix} ae & 0 \\ (b,c)e + d(f,g) & dh \end{bmatrix} = \begin{bmatrix} ae & 0 \\ (be+df, ce+dg) & dh \end{bmatrix}.$$

Now let $kQ \to K$ be the map given by extending

$$e_1 \mapsto \begin{bmatrix} 1 & 0 \\ (0,0) & 0 \end{bmatrix}, \quad e_2 \mapsto \begin{bmatrix} 0 & 0 \\ (0,0) & 1 \end{bmatrix}, \quad \alpha \mapsto \begin{bmatrix} 0 & 0 \\ (1,0) & 0 \end{bmatrix}, \quad \text{and} \quad \beta \mapsto \begin{bmatrix} 0 & 0 \\ (0,1) & 0 \end{bmatrix}$$

linearly. This sends basis elements to basis elements and respects multiplication therefore it is an isomorphism of algebras. Hence $kQ = K$.

3. Trivial paths

Fix a quiver Q. For each vertex $i \in Q_0$ we continue to let $e_i \in kQ$ be the trivial path at i. Note if p is any other path then $e_i p = p$ if $t(p) = i$ and $e_i p = 0$ otherwise. Thus multiplication on the left by e_i fixes all basis vectors whose paths end at i and kills all basis vectors whose paths do not end at i. Similarly $pe_i = p$ if $s(p) = i$ and $pe_i = 0$ otherwise so multiplication on the left by e_i fixes all basis vectors whose paths begin at i and kills all basis vectors whose paths do not begin at i.

Now it is easy to see that the left ideal Ae_i has a basis consisting of all paths beginning at i. Likewise the right ideal e_iA has a basis consisting of all paths ending at i. The linear subspace e_jAe_i has a basis consisting of all paths from i to j and the linear subspace spanned by Ae_iA has a basis consisting of all paths through i.

Proposition 1. *Let Q be a quiver and kQ its path algebra.*

1. *The trivial paths e_i are orthogonal idempotents.*
2. *The element $\sum_i e_i$ is the identity element of kQ.*
3. *The A-module Ae_i is projective.*
4. *The k-algebra e_iAe_i is a domain with identity.*
5. *For any A-module M we have $\mathrm{Hom}_A(Ae_i, M) = e_iM$ as k-vector spaces.*
6. *The Ae_i are indecomposible and inequivalent ($Ae_i \simeq Ae_j$ implies $i = j$).*

Proof.

1. That $e_i^2 = e^i$ and $e_ie_j = 0$ for all $i \neq j$ is immediate.
2. It suffices to check the claim on basis elements of kQ and on these elements the assertion is clear.
3. From (1) and (2) follows the decomposition $A = \bigoplus_i Ae_i$. Thus Ae_i is a direct summand of a free A-module.
4. Clearly e_iAe_i is closed under addition and multiplication and e_i acts as the identity. Assume $a, b \in e_iAe_i$ are neither zero nor the identity and write

each as a linear combination of paths. A single path of length $m+n>0$ cannot be written as a path of length n composed with a path of length m in more than one way. Hence choosing n and m to be the maximal lengths of paths in a and b we see that ab must contain a path of length $m+n$. Thus $ab \neq 0$.

(5) The map $\phi \mapsto \phi(e_i) = \phi(e_i^2) = e_i\phi(e_i)$ is k-linear. Any A-module morphism $\phi\colon Ae_i \to M$ is defined by the image of e_i so it is an isomorphism.

(6) One can check, in the case $M = Ae_i$, that the vector space isomorphism of (5) respects multiplication and is therefore an algebra isomorphism. If Ae_i were decomposable then a projection map $M \oplus N \to M \oplus 0$ would give a non-trivial idempotent in $\mathrm{End}_A(Ae_i) = e_iAe_i$, violating (4). Thus Ae_i is indecomposable.

Assume $\phi\colon Ae_i \to Ae_j$ and $\psi\colon Ae_j \to Ae_i$ are inverse A-module maps. Then $e_i = \psi\phi(e_i) = \psi(\phi(e_i)e_j) = \phi(e_i)\psi(e_j) = \phi(e_i)e_j\psi(e_j)$ is a path through e_j implying $i = j$.

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E-mail address: jstarx@gmail.com