FUNCTORIAL APPROACH TO ALMOST SPLIT SEQUENCES

JIM STARK

1. Some Category Theory

In this section we introduce functor k-categories and the Yoneda embedding. Fix a field k and recall that a k-category $\mathcal{C}$ is a category such that $\mathrm{Mor}_{\mathcal{C}}(M, N)$ is a k-vector space and composition is bilinear. A functor F between two k-categories is k-linear if the induced maps $\mathrm{Mor}(M, N) \to \mathrm{Mor}(FM, FN)$ are k-linear. In these notes by category we mean an additive k-category and by functor we mean a k-linear functor. We will use the notation $\mathrm{Nat}(F, T)$ for the collection of natural transformations from F to T.

Definition 1. Let $\mathcal{B}$ and $\mathcal{C}$ be two categories. Then $\mathcal{C}^{\mathcal{B}}$ is the category whose objects are all functors $F\colon \mathcal{B} \to \mathcal{C}$ and whose morphisms from F to T are the natural transformations $\mathrm{Nat}(F, T)$ with the usual identity and composition law.

Proposition 1.1. *If $\mathcal{B}$ and $\mathcal{C}$ are abelian then so is $\mathcal{C}^{\mathcal{B}}$.*

Implicit in the definition above is the claim that if $\mathcal{B}$ and $\mathcal{C}$ are additive k-categories then so is $\mathcal{C}^{\mathcal{B}}$. Instead of a proof of this fact and of the proposition above we will just give the various constructions, as well as some additional facts, below. The verification that these constructions satisfy the appropriate universal properties is straightforward in all cases.

- Let $\Phi, \Psi \in \mathrm{Nat}(F, T)$, $a, b \in k$. Then $a\Phi + b\Psi \in \mathrm{Nat}(F, T)$ is defined by
$$(a\Phi + b\Psi)_M = a\Phi_M + b\Psi_M.$$
- For any $F, T \in \mathcal{C}^{\mathcal{B}}$ the functor $F \oplus T$ is given by
$$(F \oplus T)(M) = FM \oplus TM \quad \text{and} \quad (F \oplus T)(f) = \begin{bmatrix} F(f) & 0 \\ 0 & T(f) \end{bmatrix}.$$
- The zero object in $\mathcal{C}^{\mathcal{B}}$ is the functor that maps every object in $\mathcal{B}$ to the zero object in $\mathcal{C}$.
- If $\Phi \in \mathrm{Nat}(F, T)$ then $(\ker \Phi)(X) = \ker(\Phi_X)$ for all $X \in \mathcal{B}$. The image and cokernel of a functor are defined similarly.
- A sequence $F \to T \to G$ is exact at T if and only if $FX \to TX \to GX$ is exact at T for every $X \in \mathcal{B}$.
- Subobjects of $T \in \mathcal{C}^{\mathcal{B}}$ correspond to subfunctors; that is, functors $F \in \mathcal{C}^{\mathcal{B}}$ such that FX is a subobject of TX for all $X \in \mathcal{B}$ and the inclusions give a natural transformation $F \to T$.
- A morphism $\Phi \in \mathrm{Nat}(F, T)$ is a monic, an epic, or an isomorphism if and only if Φ_X is for all $X \in \mathcal{B}$.

The Yoneda embedding is a way to embed a category into a functor category. Specifically it embeds as the full subcategory of representable functors.

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Definition 2. A *representable* functor F is any functor naturally isomorphic to $\mathrm{Mor}_{\mathcal{C}}(X, -)$. We say that X is the representing object of F.

Lemma 1.2 (Yoneda's Lemma). *Let $\mathcal{C}$ be an additive k-category, $X \in \mathcal{C}$, and $F\colon \mathcal{C} \to k-\underline{\mathrm{mod}}$ a functor. Then there is a bijection $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(X, -), F) \simeq FX$ that is functorial in X and natural in F.*

Sketch of proof. The maps are

$$\begin{aligned} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(X,-),F) &\to FX \\ \Phi &\mapsto \Phi_X(\mathrm{id}_X) \end{aligned} \qquad \text{and} \qquad \begin{aligned} FX &\to \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(X,-),F) \\ v &\mapsto [f \mapsto F(f)(v)]. \end{aligned}$$

That these maps are inverses, functorial in X, and natural in F is straightforward to check. □

We say the vector in FX *represents* the transformation associated to it under this bijection. Of particular interest is the case $F = \mathrm{Mor}_{\mathcal{C}}(Y, -)$ where the map

$$\mathrm{Mor}_{\mathcal{C}}(Y, X) \xrightarrow{\sim} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(X, -), \mathrm{Mor}_{\mathcal{C}}(Y, -))$$

is given by $f \mapsto - \circ f$. Functoriality of the Yoneda lemma then implies that if $\mathrm{Mor}_{\mathcal{C}}(X, -) \to \mathrm{Mor}_{\mathcal{C}}(Y, -)$ and $\mathrm{Mor}_{\mathcal{C}}(Y, -) \to \mathrm{Mor}_{\mathcal{C}}(Z, -)$ are represented by the morphisms $\phi \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ and $\psi \in \mathrm{Mor}_{\mathcal{C}}(Z, Y)$ respectively then their composition $\mathrm{Mor}_{\mathcal{C}}(X, -) \to \mathrm{Mor}_{\mathcal{C}}(Z, -)$ is represented by $\phi \circ \psi \in \mathrm{Mor}_{\mathcal{C}}(Z, X)$. The identity transformation is represented by the identity morphism so we have the following.

Theorem 1.3 (Yoneda Embedding). *There is an embedding $\mathcal{C}^{\mathrm{op}} \to (k-\underline{\mathrm{mod}})^{\mathcal{C}}$.*

Proof. Map an object X to $\mathrm{Mor}_{\mathcal{C}}(X, -)$. A morphism $f\colon Y \to X$ in $\mathcal{C}^{\mathrm{op}}$ is by definition a morphism $f\colon X \to Y$ in $\mathcal{C}$ so map it to the natural transformation $- \circ f\colon \mathrm{Mor}_{\mathcal{C}}(Y, -) \to \mathrm{Mor}_{\mathcal{C}}(X, -)$. We have shown above that this is functorial. The Yoneda lemma gives that it is fully faithful. □

We end this section with the dual statements about contravariant functors. A contravariant functor is representable and represented by X if it is naturally isomorphic to $\mathrm{Mor}_{\mathcal{C}}(-, X)$.

Proposition 1.4.

(i) *If $X \in \mathcal{C}$ and $F\colon \mathcal{C} \to k-\underline{\mathrm{mod}}$ is a contravariant functor then there is a bijection $\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(-, X), F) \simeq FX$ that is functorial (contravariant) in X and natural in F.*
(ii) *There is an embedding $\mathcal{C} \to (k-\underline{\mathrm{mod}})^{\mathcal{C}^{\mathrm{op}}}$.*

Proof. A contravariant functor $\mathcal{C} \to k$–$\underline{\mathrm{mod}}$ is the same thing as a covariant functor $\mathcal{C}^{\mathrm{op}} \to k-\underline{\mathrm{mod}}$ so we just apply Lemma 1.2 and Theorem 1.3 above. The maps in (1) are the same as in Lemma 1.2, they are

$$\begin{aligned} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(-,X),F) &\to FX \\ \Phi &\mapsto \Phi_X(\mathrm{id}_X) \end{aligned} \qquad \text{and} \qquad \begin{aligned} FX &\to \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(-,X),F) \\ v &\mapsto [f \mapsto F(f)(v)]. \end{aligned}$$

The functor in (2) maps an object X to $\mathrm{Mor}_{\mathcal{C}}(-, X)$ and a morphism $f\colon X \to Y$ to the natural transformation $f \circ -\colon \mathrm{Mor}_{\mathcal{C}}(-, X) \to \mathrm{Mor}_{\mathcal{C}}(-, Y)$. □

2. The Categories Fun(A) and Fun$^{\mathrm{op}}(A)$

Specify to the case $\mathcal{C} = A-\underline{\mathrm{mod}}$.

Definition 3. For any k-algebra A let

$$\mathrm{Fun}(A) = (k-\underline{\mathrm{mod}})^{A-\underline{\mathrm{mod}}} \qquad \text{and} \qquad \mathrm{Fun}^{\mathrm{op}}(A) = (k-\underline{\mathrm{mod}})^{(A-\underline{\mathrm{mod}})^{op}}$$

be the functor categories defined in the previous section. We say that an object F in either of these categories is *finitely generated* if it is a quotient of a representable functor.

In this section we establish the following classifications for $\mathrm{Fun}(A)$ and $\mathrm{Fun}^{\mathrm{op}}(A)$:

- The finitely generated projective objects are the representable functors.
- Such a functor is indecomposable if and only if its representing object is.
- Isomorphism classes of simple objects are in bijective correspondence with isomorphism classes of indecomposable A-modules. The correspondence maps M to the equivalence class of the functor

$$S_M = \mathrm{Hom}_A(M,-)/\mathrm{rad}_A(M,-) \qquad (\text{in } \mathrm{Fun}(A))$$
$$S^M = \mathrm{Hom}_A(-,M)/\mathrm{rad}_A(-,M) \qquad (\text{in } \mathrm{Fun}^{\mathrm{op}}(A)).$$

To begin we have the Yoneda embeddings

$$(A-\underline{\mathrm{mod}})^{op} \to \mathrm{Fun}(A) \qquad \text{and} \qquad A-\underline{\mathrm{mod}} \to \mathrm{Fun}^{\mathrm{op}}(A).$$

Recall that an object P in either $\mathrm{Fun}(A)$ or $\mathrm{Fun}^{\mathrm{op}}(A)$ is projective if, given any natural transformation $\Phi\colon F \to G$, the map $\Phi \circ -\colon \mathrm{Nat}(P,F) \to \mathrm{Nat}(P,G)$ is surjective. We take care of the first two points in the above classification with the following proposition.

Proposition 2.1. *The embedding* $(A-\underline{\mathrm{mod}})^{op} \to \mathrm{Fun}(A)$ *induces an equivalence of categories between* $(A-\underline{\mathrm{mod}})^{op}$ *and* $\mathrm{fgp}\,\mathrm{Fun}(A)$*, the full subcategory of finitely generated projective objects in* $\mathrm{Fun}(A)$*. Moreover if* $F \in \mathrm{fgp}\,\mathrm{Fun}(A)$ *is represented by* M *then* F *is indecomposable in* $\mathrm{Fun}(A)$ *if and only if* M *is an indecomposable* A*-module.*

Proof. We already know that the embedding is fully faithful so to prove that it is an equivalence of categories we must show that its image is a dense subcategory of $\mathrm{fgp}\,\mathrm{Fun}(A)$. First we claim that each $\mathrm{Hom}_A(M,-)$ is a projective object. Let $\Phi\colon F \to G$ be an epimorphism. Naturality of Yoneda's lemma gives the commutative diagram below.

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{Hom}_A(M,-),F) & \xrightarrow{\sim} & FM \\ \downarrow{\scriptstyle \Phi\circ -} & & \downarrow{\scriptstyle \Phi_M} \\ \mathrm{Nat}(\mathrm{Hom}_A(M,-),G) & \xrightarrow{\sim} & GM \end{array}$$

By hypothesis Φ_M is surjective and the other two maps are bijections therefore $\Phi \circ -$ is surjective. This proves that $\mathrm{Hom}_A(M,-)$ is projective.

Now we prove that the image is dense in $\mathrm{fgp}\,\mathrm{Fun}(A)$, so let F be a projective object and $\Phi\colon \mathrm{Hom}_A(M,-) \to F$ an epimorphism (so that F is finitely generated). As F is projective there exists a splitting $\Psi\colon F \to \mathrm{Hom}_A(M,-)$ and $\Psi\Phi \in \mathrm{End}(\mathrm{Hom}_A(M,-))$ is an idempotent. By Yoneda's lemma $\Psi\Phi$ is represented by some idempotent $f \in \operatorname{End}(M)$ therefore $M = \operatorname{im} f \oplus \ker f$. Additive functors split over direct sums therefore

$$\operatorname{Hom}_A(M, -) \simeq \operatorname{Hom}_A(\operatorname{im} f, -) \oplus \operatorname{Hom}_A(\ker f, -).$$

But note that $\Psi\Phi = - \circ f$ and f idempotent means it is the identity on its image. Therefore we conclude that $\operatorname{Hom}_A(\operatorname{im} f, -) \simeq \operatorname{im}(\Psi\Phi) \simeq F$ is representable and the embedding is dense as desired.

In any abelian category a direct summand of a projective object is projective and by Yoneda the representing object of F is uniquely determined up to isomorphism therefore $\operatorname{Hom}_A(M \oplus N, -) \simeq \operatorname{Hom}_A(M, -) \oplus \operatorname{Hom}_A(N, -)$ implies that a representable functor is indecomposable if and only if its representing object is. □

To classify the simple objects we first prove that the S_M are in fact simple.

Lemma 2.2. *If M is an indecomposable A-module then $\operatorname{rad}_A(M, -)$ is the unique maximal subfunctor of $\operatorname{Hom}_A(M, -)$.*

Proof. Clearly $\operatorname{rad}_A(M, -)$ is a proper subobject because $\operatorname{rad}_A(M, M)$ doesn't contain the identity map therefore it suffices to show that every proper subfunctor $F \subseteq \operatorname{Hom}_A(M, -)$ is a subfunctor of $\operatorname{rad}_A(M, -)$. Being additive functors both $\operatorname{Hom}_A(M, -)$ and $\operatorname{rad}_A(M, -)$ split over direct sums so it suffices to show that $F(N) \subseteq \operatorname{rad}_A(M, N)$ when N is an indecomposable A-module. If $N \not\simeq M$ then $\operatorname{rad}_A(M, N) = \operatorname{Hom}_A(M, N)$ and the conclusion is trivial so all that remains is to show that $F(M) \subseteq \operatorname{rad}_A(M, M)$.

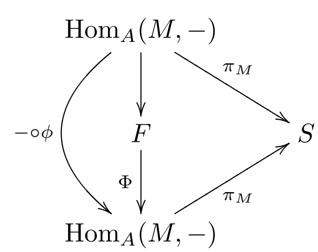
As F is a subfunctor of $\operatorname{Hom}_A(M, -)$ we have $F(f) = f \circ -$. Assume $\phi \in F(M)$. By Yoneda's lemma $- \circ \phi$ defines a natural transformation $\operatorname{Hom}_A(M, -) \to F$, hence a natural transformation $\operatorname{Hom}_A(M, -) \to F \to \operatorname{Hom}_A(M, -)$. As F is a proper subfunctor this natural transformation is not an isomorphism therefore Yoneda implies that ϕ is not an isomorphism either. Hence $\phi \in \operatorname{rad}_A(M, M)$ as desired. □

Clearly this is well defined as a mapping of isomorphism classes. Part 2 of the following implies that it is a bijection.

Proposition 2.3. *Let $S \in \operatorname{Fun}(A)$ be a simple object.*

(i) *If M is indecomposable and $\pi_M \colon \operatorname{Hom}_A(M, -) \to S$ is nonzero then for any Φ if $\pi_M \circ \Phi$ is an epimorphism then so is Φ.*
(ii) *There is a unique indecomposable A-module M such that $S(M) \neq 0$. For such an M we have $S \simeq S_M$ and $S(X) \neq 0$ if and only if M is a direct summand of X.*

Proof. Assume we have $\Phi \colon F \to \operatorname{Hom}_A(M, -)$ such that $\pi_M \circ \Phi \colon F \to S$ is an epimorphism. Then because $\operatorname{Hom}_A(M, -)$ is projective there is a map $\operatorname{Hom}_A(M, -) \to F$ making the diagram below commute.



The composition $\mathrm{Hom}_A(M,-) \to F \to \mathrm{Hom}_A(M,-)$ is represented by some element $\phi \in \mathrm{End}(M)$. As M is indecomposable $\mathrm{End}(M)$ is local so ϕ is either nilpotent or an isomorphism. Commutativity of the diagram and π_M nonzero shows that ϕ is not nilpotent therefore it must be an isomorphism. The composition is represents is therefore an isomorphism so Φ is an epimorphism. This proves (i).

For (ii) we know S is nonzero therefore $S(M) \neq 0$ for some M. As S splits over direct sums we can assume M is indecomposable. By Yoneda there exists a nonzero map $\pi_M \colon \mathrm{Hom}_A(M,-) \to S$. As S is simple the kernel of π_M is a maximal subobject in $\mathrm{Hom}_A(M,-)$, so by Lemma 2.2 the kernel is $\mathrm{rad}_A(M,-)$ and $S \simeq S_M$.

Assume that $S(X) \neq 0$ for some X and let $\pi_X \colon \mathrm{Hom}_A(X,-) \to S$ nonzero be given by Yoneda as well. As above these are epimorphisms so $\mathrm{Hom}_A(X,-)$ projective implies there is a map $\Phi \colon \mathrm{Hom}_A(X,-) \to \mathrm{Hom}_A(M,-)$ such that $\pi_X = \pi_M \circ \Phi$. By Part (i) the map Φ is an epimorphism so $\mathrm{Hom}_A(M,-)$ projective implies $\mathrm{Hom}_A(M,-)$ is a direct summand of $\mathrm{Hom}_A(X,-)$. As in the proof of Proposition 2.1 this implies M is a direct summand of X. In particular this shows that M is unique. □

Last but not least we state the dual results for $\mathrm{Fun}^{\mathrm{op}}(A)$.

Proposition 2.4.

(i) *The embedding* $A-\underline{\mathrm{mod}} \to \mathrm{Fun}^{\mathrm{op}}(A)$ *induces an equivalence of categories between* $A-\underline{\mathrm{mod}}$ *and* $\mathrm{fgp}\,\mathrm{Fun}^{\mathrm{op}}(A)$. *Moreover if* $F \in \mathrm{fgp}\,\mathrm{Fun}^{\mathrm{op}}(A)$ *is represented by* M *then* F *is indecomposable in* $\mathrm{Fun}^{\mathrm{op}}(A)$ *if and only if* M *is an indecomposable* A*-module.*

(ii) *If* M *is an indecomposable* A*-module then* $\mathrm{rad}_A(-,M)$ *is the unique maximal subfunctor of* $\mathrm{Hom}_A(-,M)$.

(iii) *If* $\pi^M \colon \mathrm{Hom}_A(-,M) \to S^M$ *is the canonical quotient map then for any* Φ *if* $\pi^M \circ \Phi$ *is an epimorphism then so is* Φ.

(iv) *Let* $S \in \mathrm{Fun}^{\mathrm{op}}(A)$ *be a simple object. Then there is a unique indecomposable* A*-module* M *such that* $S(M) \neq 0$. *For such an* M *we have* $S \simeq S^M$ *and* $S(X) \neq 0$ *if and only if* M *is a direct summand of* X.

3. Projective presentations and almost split sequences

In $\mathrm{Fun}(A)$ and $\mathrm{Fun}^{\mathrm{op}}(A)$ we have projective objects and epimorphisms. In order to define projective covers and minimal projective presentations we need only define the notion of a minimal epimorphism. The remaining definitions are the same as in $A-\underline{\mathrm{mod}}$.

Definition 4. An epimorphism $\Phi \in \mathrm{Nat}(F,G)$ is *minimal* if for every H and every $\Psi \in \mathrm{Nat}(H,F)$ it is the case that $\Phi \circ \Psi$ an epimorphism implies Ψ is an epimorphism. In general a map is minimal if it is a minimal epimorphism onto its image.

Corollary 3.1. *The maps* $\pi_M \colon \mathrm{Hom}_A(M,-) \to S_M$ *and* $\pi^M \colon \mathrm{Hom}_A(-,M) \to S^M$ *are projective covers.*

We will now start to link minimal projective presentations in $\mathrm{Fun}(A)$ and $\mathrm{Fun}^{\mathrm{op}}(A)$ to almost split sequences in $A-\underline{\mathrm{mod}}$.

Lemma 3.2.

(i) *Let L be an indecomposable module. A map $f\colon L \to M$ is left almost split if and only if the image of $- \circ f\colon \mathrm{Hom}_A(M,-) \to \mathrm{Hom}_A(L,-)$ is $\mathrm{rad}_A(L,-)$.*
(ii) *Let N be an indecomposable module. A map $g\colon M \to N$ is right almost split if and only if the image of $g \circ -\colon \mathrm{Hom}_A(-,M) \to \mathrm{Hom}_A(-,N)$ is $\mathrm{rad}_A(-,N)$.*

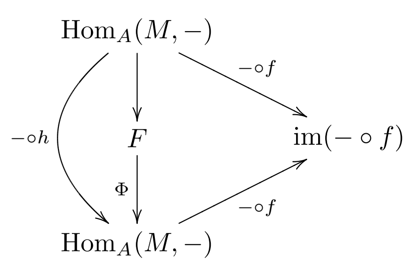
Proof. We prove (i), the proof of (ii) is similar. Images are pointwise and functors split over direct sums so the condition on $-\circ f$ is equivalent to the condition that for every indecomposable module X the image of $-\circ f\colon \mathrm{Hom}_A(M,X) \to \mathrm{Hom}_A(L,X)$ is $\mathrm{rad}_A(L,X)$, the non-invertible maps $L \to X$. Assume this is the case. Then f is not a section because $h \circ f \in \mathrm{rad}_A(L,L)$ is always non-invertible. The lifting property can be shown for indecomposable modules so assume X is indecomposable and $u\colon L \to X$ is not a section. Then it is not invertible so $u \in \mathrm{rad}_A(L,X) = \mathrm{im}(-\circ f)$ which means h lifts as desired. This proves that f is left almost split.

Next assume f is left almost split and X is an indecomposable module. A section between two indecomposable modules is an isomorphism so each $h \in \mathrm{rad}_A(L,X)$ is not a section and f left almost split implies $h \in \mathrm{im}(-\circ f)$. Conversely $X \not\simeq L$ implies $\mathrm{Hom}_A(L,X) = \mathrm{rad}_A(L,X)$ and for $X \simeq L$ the fact that f is not a section implies $h \circ f$ is never invertible, so in either case $\mathrm{im}(-\circ f) \subseteq \mathrm{rad}_A(L,X)$. □

Lemma 3.3.

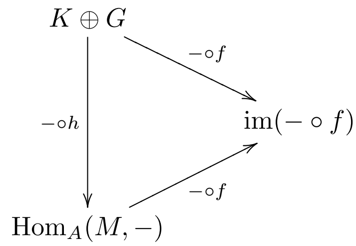
(i) *Let L be an indecomposable module. A map $f\colon L \to M$ is left minimal if and only if the induced map $-\circ f\colon \mathrm{Hom}_A(M,-) \to \mathrm{Hom}_A(L,-)$ is minimal.*
(ii) *Let N be an indecomposable module. A map $g\colon M \to N$ is right minimal if and only if the induced map $g \circ -\colon \mathrm{Hom}_A(-,M) \to \mathrm{Hom}_A(L,-)$ is minimal.*

Proof. We prove (i), the proof of (ii) is similar. First assume f is left minimal and $\Phi\colon F \to \mathrm{Hom}_A(M,-)$ is natural such that $(-\circ f)\circ \Phi$ is an epimorphism. We proceed as in the proof of Proposition 2.3, projectivity gives the diagram below



and Yoneda gives h. The bijectivity of Yoneda gives $hf = f$ so h, and hence the natural transformation it represents, is an isomorphism. Thus Φ is surjective.

Now assume $-\circ f$ is minimal and $h \in \mathrm{End}(M)$ such that $hf = f$. Minimality and the diagram above imply that $-\circ h\colon \mathrm{Hom}_A(M,-) \to \mathrm{Hom}_A(M,-)$ is surjective. By projectivity the kernel K is a direct summand of $\mathrm{Hom}(M,-)$ so the diagram is the following:



Commutativity implies that the inclusion $G \to K \oplus G$ followed by $- \circ f$ is surjective so $G \to K \oplus G$ is surjective. Hence $- \circ h$ is injective and therefore an isomorphism. By Yoneda h is an isomorphism. □

Now we have the tools we need to present the main theorem.

Main Theorem.

- *Let L be an indecomposable module.*
  - (i) *L is injective and $f\colon L \to M$ is left minimal almost split if and only if*

$$0 \to \mathrm{Hom}_A(M, -) \overset{-\circ f}{\to} \mathrm{Hom}_A(L, -) \overset{\pi_L}{\to} S_L \to 0 \qquad (*)$$

    is a minimal projective resolution of S_L in $\mathrm{Fun}(A)$.
  - (ii) *L is not injective and $0 \to L \overset{f}{\to} M \overset{g}{\to} N \to 0$ is an almost split exact sequence if and only if $N \neq 0$ and*

$$0 \to \mathrm{Hom}_A(N, -) \overset{-\circ g}{\to} \mathrm{Hom}_A(M, -) \overset{-\circ f}{\to} \mathrm{Hom}_A(L, -) \overset{\pi_L}{\to} S_L \to 0 \qquad (**)$$

    is a minimal projective resolution of S_L in $\mathrm{Fun}(A)$.
- *Let N be an indecomposable module.*
  - (i) *N is projective and $g\colon M \to N$ is right minimal almost split if and only if*

$$0 \to \mathrm{Hom}_A(-, M) \overset{g\circ -}{\to} \mathrm{Hom}_A(-, N) \overset{\pi^N}{\to} S^N \to 0$$

    is a minimal projective resolution of S_L in $\mathrm{Fun}^{\mathrm{op}}(A)$.
  - (ii) *N is not projective and $0 \to L \overset{f}{\to} M \overset{g}{\to} M \to 0$ is an almost split exact sequence if and only if $L \neq 0$ and*

$$0 \to \mathrm{Hom}_A(-, L) \overset{f\circ -}{\to} \mathrm{Hom}_A(-, M) \overset{g\circ -}{\to} \mathrm{Hom}_A(-, N) \overset{\pi^N}{\to} S^N \to 0$$

    is a minimal projective resolution of S_L in $\mathrm{Fun}^{\mathrm{op}}(A)$.

Proof. We prove the covariant case, the contravariant case is similar. First note that Lemmas 3.2 and 3.3 imply that f is left minimal almost split if and only if

$$\mathrm{Hom}_A(M, -) \overset{-\circ f}{\to} \mathrm{Hom}_A(L, -) \overset{\pi_L}{\to} S_L \to 0$$

is a minimal projective presentation of S_L. For (i) assume this is the case and L is injective, we need that $- \circ f$ is a monomorphism. But Proposition 3.5 in the text gives that f is an epimorphism so left exactness of Hom_A implies that the induced map $- \circ f$ is a monomorphism as desired.

Conversely if $(*)$ is a minimal projective resolution then f is left minimal almost split by the above reasoning, we need that L is injective. Evaluating at $\mathrm{coker}\, f$ gives an injective map $- \circ f\colon \operatorname{Hom}_A(M, \operatorname{coker} f) \to \operatorname{Hom}_A(L, \operatorname{coker} f)$. The standard projection $M \to \operatorname{coker} f$ is the zero map because both are sent to the zero map under $- \circ f$. This gives $\operatorname{coker} f = 0$ so f is surjective, hence not the left map in an almost split short exact sequence, hence L is injective.

For (ii) assume L is not injective and $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an almost split short exact sequence. Then f is left minimal almost split so the minimal projective presentation is as above. Left exactness of Hom_A them implies that $(**)$ is the minimal projective resolution. By uniqueness of almost split exact sequences $L \simeq \tau^{-1}L \neq 0$ because L is not injective.

Conversely assume $N \neq 0$ and $(**)$ is the minimal projective resolution. Then f is left minimal almost split. If L were injective then by (i) the map $- \circ f$ would be injective and therefore $\operatorname{Hom}_A(N, -) = 0$. But then $\operatorname{Hom}_A(N, N) = 0$ and this is a contradiction, because $N \neq 0$ implies that the identity map is distinct from the zero map. So L is not injective and f is left minimal almost split. Proposition 3.1 in the text (and uniqueness of minimal almost split morphisms) gives an almost split short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g'} \tau^{-1}L \to 0$. The previous direction then implies that $\operatorname{Hom}_A(\tau^{-1}L, -) \simeq \operatorname{Hom}_A(N, -)$. In particular we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_A(N, -) & \xrightarrow{\sim} & \operatorname{Hom}_A(\tau^{-1}L, -) \\ \big\downarrow{\scriptstyle -\circ g} & \swarrow{\scriptstyle -\circ g'} & \\ \operatorname{Hom}(M, -) & & \end{array}$$

so by Yoneda we have a commutative diagram

$$\begin{array}{ccc} N & \xleftarrow{\sim} & \tau^{-1}L \\ {\scriptstyle g}\big\uparrow & \nearrow{\scriptstyle g'} & \\ \operatorname{Hom}(M, -) & & \end{array}$$

therefore $0 \to L \to M \to N \to 0$ is an almost split short exact sequence. $\square$

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E-mail address: jstarx@gmail.com