Examples of Regular Modules over a Wild Algebra

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1 Results

Throughout this notes, A is a basic and connected finite dimensional K-algebra with K algebraically closed.

Defninition 1.1 (Auslander-Reiten Quiver)**.** *The* <u>*AR quiver*</u> $\Gamma(\mathrm{mod}A)$ *of* A *is defined as follows.*

- *The vertices of* Γ *are the isomorphism classes* $[X]$ *of indecomposable modules* X *in* $\mathrm{mod}A$ *and*
- *Let* $[X], [Y]$ *be the points in* Γ *corresponding to the indecomposable modules* $X, Y \in \mathrm{mod}A$*. The arrows* $[X] \to [Y]$ *are in bijective corresponding to the vectors of a basis of the* K*-vector space Irr*(XY)*.*

Lemma 1.2 (A criterion for almost split sequences)**.**

1. *Let* M *be an indecomposable nonprojective* A*-module. If* $\underline{\mathrm{End}}M$ *is a skew field, then any nonsplit short exact sequence*
$$0 \to \tau M \to E \to M \to 0$$
is almost split and $\underline{\mathrm{End}}M \cong K$*.*

2. *Let* N *be an indecomposable noninjective* A*-module. If* $\overline{\mathrm{End}}N$ *is a skew filed, then any nonsplit exact sequence*
$$0 \to N \to F \to \tau^{-1}N \to 0$$
is almost split sequence and $\overline{\mathrm{End}}N \cong K$*.*

***Proof*.** [1] Corollary IV.3.2.

Lemma 1.3. 1. *Let P be an indecomposable projective A-module. An A-module homomorphism $g : M \to P$ is right minimal almost split if and only if g is a monomorphism and $M \cong \mathrm{rad}P$.*

2. *Let I be an indecomposable injective A-module. An A-module homomorphism $f : I \to M$ is left minimal almost split if and only if f is an epimorphism and $M \cong I/\mathrm{soc}I$*

Proof. [1] Proposition IV.3.5.

Lemma 1.4. *[Simple projective noninjective]*

1. *Let S be a simple projective noninjective A-module. If $S \to M$ is irreducible, then M is projective.*
2. *Let S be a simple injective nonprojective A-module, then $M \to S$ is irreducible if and only if M is injective.*

Proof. [1] Corollary IV.3.9.

Remark 1.5. *Let S be a simple projective-noninjective. It follows from Lemma 1.4 that $0 \to S \xrightarrow{f} P \to \mathrm{coker}f \to 0$ is almost split then P is the direct sum of all indecomposable projective A-modules P' such that S is isomorphic to a direct summand of $\mathrm{rad}P'$. Similarly, if S is a simple injective-nonprojective, then $0 \to \ker g \to I \xrightarrow{g} S \to 0$ is almost split then I is the direct sum of all indecomposable injective A-modules I' such that S is isomorphic to a direct summand of $I'/\mathrm{soc}I'$.*

Lemma 1.6. *[projective-injective-nonsimple] Let P be a nonsimple indecomposable projective injective A-module, $S = \mathrm{soc}P, R = \mathrm{rad}P$. Then the sequence*

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} q \\ i \end{smallmatrix}\right]} R/S \oplus P \xrightarrow{[-j\ p]} P/S \longrightarrow 0$$

is almost split, where i, j are the inclusions and $p.q$ the projections.

Proof. [1] IV.3.11.

Proposition 1.7. *Let A be a representation-finite algebra. Then $\Gamma(\mathrm{mod}A)$ has no multiple arrows.*

Proof. [1] IV.4.9.

2 Examples

Example 2.1. Let A be the path algebra of the quiver

$$1 \xleftarrow{\beta} 2 \xleftarrow{\alpha} 3 .$$

We list the indecomposable projective and injective A-modules as follows.

$$\begin{aligned} P(1) &= (K \longleftarrow 0 \longleftarrow 0) = S(1) \\ P(2) &= (K \longleftarrow K \longleftarrow 0) \\ P(3) &= (K \longleftarrow K \longleftarrow K) = I(1) \\ I(2) &= (0 \longleftarrow K \longleftarrow K) \\ I(3) &= (0 \longleftarrow 0 \longleftarrow K). \end{aligned}$$

First of all, because $P(1)$ is simple projective noninjective and $P(1) = \mathrm{rad}P(2)$, by Lemma 1.4 and its remarks we have the almost split sequence

$$0 \longrightarrow P(1) \xrightarrow{i} P(2) \longrightarrow \mathrm{coker}i \longrightarrow 0.$$

It is easy to see that $\mathrm{coker}i \cong P(2)/P(1) \cong S(2)$.

Secondly note that $P(3)$ is nonsimple projective injective module. By Lemma 1.6, we have the following almost split sequence

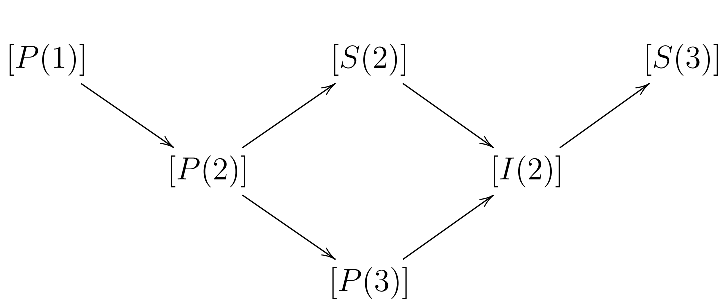
$$0 \longrightarrow P(2) \longrightarrow P(3) \oplus S(2) \longrightarrow I(2) \longrightarrow 0.$$

Finally, since $S(3)$ is simple injective nonprojective module, again by Lemma 1.4, we have the the almost split sequence

$$0 \longrightarrow \ker j \longrightarrow I(2) \xrightarrow{j} S(3) \longrightarrow 0.$$

It is easy to see that $\ker j \cong S(2)$.

Putting together all the information we obtained, $\Gamma(\mathrm{mod}A)$ is the quiver



Example 2.2.

References

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