Homework 3 for Group Cohomology, MATH 682
due March 8

Problem 1. (" 5 term exact sequence")
(a). Let $E^2_{pq} \Rightarrow H_{p+q}(C)$ be a homology spectral sequence of a double complex C concentrated in the first quadrant. show that there is a 5 term exact sequence

$$H_2(C) \to E^2_{20} \xrightarrow{d} E^2_{01} \to H_1(C) \to E^2_{10} \to 0.$$

(b). Write down and prove the corresponding exact sequence for cohomology.

Problem 2. Prove Universal Coefficient Theorem for cohomology: if R is a PID, C_* is a chain complex which is dimension-wise flat and M is an R-module, then there is a short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C), M) \to H^n(\mathrm{Hom}_R(C, M)) \to \mathrm{Hom}_R(H_n(C), M) \to 0.$$

In fact, this sequence is non-canonically split, but you don't have to prove that!

Problem 3. Construct the "base-change" spectral sequence for Tor :
Let $f : R \to S$ be a ring map, M be an S module, and N be an R-module. Then there is a first quadrant homology spectral sequence

$$E^2_{pq} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(M, S), N) \Rightarrow \mathrm{Tor}^R_{p+q}(M, N),$$

where M is viewed as an R module via the pull-back via f.

Problem 4. Compute cohomology of S_3, the group of permutations on 3 elements.

Problem 5. Let D_m denote the dihedral group, which is the group of symmetries of the regular $m - gon$.
(a). Show that $D_m = C_m \rtimes C_2$ where the action of C_2 on C_m is via $\sigma(a) = a^{-1}$ for $a \in C_m$, and σ being a generator of C_2. Here, C_m denote a cyclic group of order m which I don't dare writing as $\mathbb{Z}/m$ now:).
(b). Since C_m is a normal subgroup in D_m, the factor group C_2 acts on cohomology $H^*(C_m, \mathbb{Z})$. Let t be a generator of C_m, σ be a generator of C_2 and $g = (1, \sigma) \in D_m$. Show that $gtg^{-1} = t^{-1}$. Then show that the map induced by congujation by g on $H^{2n}(C_m, \mathbb{Z})$ is multiplication by $(-1)^n$.
(c). Compute cohomology of D_m using LHS spectral sequence.

Problem 6. Let $\mathbf{Ch}_{\geq 0}(Ab)$ be the category of complexes of abelian groups concentrated in non-negative degrees. Show that the left hyper derived functors $\mathbb{L}_i F$ restricted to $\mathbf{Ch}_{\geq 0}(Ab)$ are the left derived functors of the right exact functor $H_0 F$ (where F is a right exact functor on Ab itself).

Problem 7. Let R be a ring, A_*, B_* be bounded below chain complexes of R-modules.
(a). Show that there exist bounded below complexes P_*, Q_* such that
- P_n, Q_n are projective for all n for which they are non-trivial,
- P_*, Q_* are quasi-isomorphic to A_*, B_* (respectively).

(Hint: consider the total complex of the Cartan-Eilenberg resolution.)

(b). If P_*, Q_* are as in (a), show that there are isomorphisms

$$H_*(A \otimes B) \simeq H_*(A \otimes Q) \simeq H_*(P \otimes B) = H_*(P \otimes Q).$$