Homework 1 for Group Cohomology, MATH 682

Unless specified otherwise, k is a field.

Problem 0. Find as many (non-identical) explanations as you can to justify that $H^i(G, kG) = 0$ for $i > 0$.

Problem 1. Let $P(M)$ be the projective cover of a G-module M and S be a simple G-module. Show that

$$Hom_G(P(M), S) = Hom_G(M, S).$$

Problem 2. Let M be a finite dimensional G module, and $M^{\#}$ be the linear dual of M, $M^{\#} = \text{Hom}_k(M, k)$.

(a) Show that M is a direct summand of $M \otimes M^{\#} \otimes M$.

(b) Conclude that the following are equivalent:

(1) M is projective
(2) $\text{End}_k(M, M) = M \otimes M^{\#}$ is projective
(3) $M^{\#}$ is projective
(4) M is injective

(c) Conclude that kG is a self-injective algebra, i.e. is an injective module over itself.

(d) Give an example of a ring R which is not an injective module over itself.

Problem 3. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of G-modules. Determine explicitly the connecting homomorphism in the long exact sequence induced in cohomology,

$$\delta : H^0(G, M_3) \to H^1(G, M_1).$$

For the next two problems, denote by I the augmentation ideal of kG, i.e. $I = \text{Ker}\{\epsilon : kG \to k\}$

Problem 4. Show that

$$\text{Ext}^1_G(k, k) \simeq \text{Hom}_k(I/I^2, k).$$

Problem 5. Prove that $H_1(G, \mathbb{Z}) = G^{ab}$ using short exact sequence of G-modules $0 \to I \to \mathbb{Z}G \to \mathbb{Z} \to 0$.

Problem 6. Compute $H^i(\mathbb{Z}/p, \mathbb{Z}/p)$ and $H_i(\mathbb{Z}/p, \mathbb{Z}/p)$.

Problem 7. Compute $H^i(\mathbb{Z}/3 \times \mathbb{Z}/2, \mathbb{Z})$ and $H_i(\mathbb{Z}/3 \times \mathbb{Z}/2, \mathbb{Z})$.

Problem 8. Show that $H^1(G, \mathbb{Z}) = 0$ for any finite group G.

Problem 9. (Hilbert's theorem 90: multiplicative version). Let L/K be a finite Galois extension of fields, with Galois group G. G clearly acts on the group of units of L, L^*. *Hilbert's theorem 90* states that if $\theta : G \to L^*$ is a derivation such that $\theta(gh) = g\theta(h)\theta(h)$, then $\theta(g) = (gx)/x$.

(a) Show that *Hilbert's theorem 90* reformulates as

$$H^1(G, L^*) = 0.$$

(b) Show that $H^2(G, L^*)$ does not necessarily vanish. (Hint: look, for example , at $\mathbb{C}/\mathbb{R}$).