Homework II for Group Cohomology, Winter 2006
due Monday, February 27

Unless specified otherwise, k is a field, G is a finite group.

Problem 0. Find a new justification for the equality $H^i(G, kG) = 0$ for $i > 0$.

Problem 1. Let Δ be a category with objects $\{[n]\}, n \geq 0$, the ordered sets of numbers $0 \dots n$, and with morphisms being the order-preserving maps. Show that any morphism is a composition of "face" and "degeneracy" maps.

Problem 2.
(a) Let $0 \to A \to \mathcal{G} \xrightarrow{p} G \to 1$ be an extension of G by an abelian group A. Show that the following are equivalent

(1) p admits a section (i.e. there is a group homomorphism $s : G \to \mathcal{G}$ such that $p \circ s = id_G$)
(2) $\mathcal{G} \simeq G \ltimes A$

(b) Let $0 \to A \to \mathcal{G} \xrightarrow{p} G \to 1$ be a fixed split extension. Two sections are called equivalent if they differ by conjugation of an element from A. Show that there is $1-1$ correspondence between $H^1(G, A)$ and the set of equivalence classes of sections of p.

Problem 3. Let $0 \to A \to \mathcal{G} \xrightarrow{p} G \to 1$ be an extension of G by an abelian group A, and let $f \in Z^2(G, A)$ be a *normalized* cocycle (i.e. $f(1, g) = f(g, 1) = 0$ for any $g \in G$). Show that $gf(g^{-1}, g) = f(g, g^{-1})$.

Problem 4. Let k be a field and n be a positive integer. Let γ denote the class in $H^2(PGL_n(k), k^*)$ corresponding to the extension $1 \to k^* \to GL_n(k) \to PGL_n(k) \to 1$. If $\rho : G \to PGL_n(k)$ is a projective representation, show that ρ lifts to a linear representation $\tilde{\rho} : G \to GL_n(k)$ if and only if $\rho^*(\gamma) = 0$ as an element of $H^2(G, k^*)$ where $\rho^* : H^*(PGL_n(k), k^*) \to H^*(G, k^*)$ is the map in cohomology induced by ρ.

Problem 5. Let G be a finite group, H be a subgroup of G, and k be a field. Prove "tensor identity": for a G-module M and an H-module N there is a canonical isomorphism

$$\mathrm{Ind}_H^G(N \otimes_k M) \simeq \mathrm{Ind}_H^G N \otimes_k M$$

Problem 6. Let $i : H \hookrightarrow G$ be a subgroup of G, M be a G-module, and $f : M \to \mathrm{Coind}_H^G M$ be the canonical map. Show that the induced map in cohomology $i^* : H^*(G, M) \to H^*(H, M)$ coincides with the map $H^*(G, M) \to H^*(G, \mathrm{Coind}_H^G M) \overset{Frob}{\simeq} H^*(H, M)$.

Problem 7. Consider $\mathbb{Z}/m\mathbb{Z}$ as a subgroup in $\mathbb{Z}/(mn)\mathbb{Z}$. Compute the associated restriction and corestriction maps on cohomology with trivial coefficients $\mathbb{Z}$.

Problem 8. Prove "double coset formula":

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G M = \bigoplus_{x \in K \backslash G / H} \mathrm{Ind}_{K \cap xHx^{-1}}^K \mathrm{Res}_{K \cap xHx^{-1}}^{xHx^{-1}} xM$$

where $K, H \subset G$ are subgroups of finite index, M is a G-module and $K \backslash G / H$ is a set of double coset representatives.