THE STABLE CATEGORY OF A FROBENIUS CATEGORY IS TRIANGULATED

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1. Exact Categories

Let $\mathcal{A}$ be an additive category. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* if f is a kernel of g and g is a cokernel of f. Note: f is monic and g is epi since kernels and cokernels are monic and epi respectively. A *morphism* of exact sequences is a triple of morphisms (φ, ψ, θ) such that

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\theta} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' \end{array}$$

commutes. A morphism (φ, ψ, θ) is an isomorphism if the three morphisms are isomorphisms.

Definition 1. Let $\mathcal{A}$ be an additive category and $\mathcal{E}$ a class of exact sequences in $\mathcal{A}$. Given an exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{E}$, f is called an *inflation* (*admissible monomorphism*) and g is a *deflation* (*admissible epimorphism*). An exact sequence in $\mathcal{E}$ will be called a *conflation* and will sometimes be denoted as (f, g). The pair $(\mathcal{A}, \mathcal{E})$ will be called *exact* (or, more breifly, $\mathcal{A}$ is exact) if the following axioms hold:

(1) Any sequence $A \to B \to C$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$.
(2) For any pair of objects A, B in $\mathcal{A}$, the canonical sequence

$$A \xrightarrow{\iota_A} A \oplus B \xrightarrow{\pi_B} B$$

is in $\mathcal{E}$.
(3) The composition of two deflations is again a deflation.
(4) The composition of two inflations is again an inflation.
(5) If $f : A \to C$ is a deflation and $g : B \to C$ is any map then the pullback exists

$$\begin{array}{ccc} A \oplus_C B & \xrightarrow{f'} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

with f' a deflation.

(6) If $f : C \to A$ is an inflation and $g : C \to B$ is any map, then the pushout exists

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ B & \xrightarrow[f']{} & B \oplus^C A \end{array}$$

with f' an inflation.

(7) Let $f : B \to C$ be a map with a kernel. If there is a map $g : A \to B$ such that $fg : A \to C$ is a deflation then f is a deflation.

(8) Let $f : A \to B$ be a map with a cokernel. If there is a map $g : B \to C$ such that gf is an inflation then f is an inflation.

Remark. Bernhard Keller [1] has shown $(\mathcal{A}, \mathcal{E})$ is exact if and only if the identity map of the zero object is a deflation and (1), (3), (4), (5) and (6) above hold.

In any additive category, $A \oplus 0 \cong A$ and we have the canonical exact sequences $A = A \to 0$ and $0 \to A = A$. Hence, identities are always conflations and deflations. For any morphism $f : A \to B$ we get commutative diagrams

$$\begin{array}{ccccc} A & = & A & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle f} & & \downarrow \\ A & \xrightarrow[f]{} & B & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} 0 & \longrightarrow & A & = & A \\ \downarrow & & \| & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & A & \xrightarrow[f]{} & B. \end{array}$$

with the top rows conflations. Hence, if $f : A \to B$ is an isomorphism then the bottom rows are also conflations by (1) of the definition of an exact category.

Example 1. (1) Any abelian category is canonically an exact category by letting $\mathcal{E}$ be the class of all short exact sequences.

(2) Let $\mathcal{A}$ be an abelian category. Let $\mathcal{E}$ be the class of all split exact sequences. Then $(\mathcal{A}, \mathcal{E})$ is an exact category.

(3) If $(\mathcal{A}, \mathcal{E})$ is an exact category then $(\mathcal{A}^{op}, \mathcal{E}^{op})$ is an exact category where $\mathcal{E}^{op}$ is the class of all sequence $C^{op} \to B^{op} \to A^{op}$ such that $A \to B \to C$ is in $\mathcal{E}$.

Example 2. Let $\mathbf{Ch}(\mathcal{A})$ be the category of chain complexes over an abelian category. Let $\mathcal{E}$ be the class of all sequences

$$0 \longrightarrow A. \xrightarrow{f.} B. \xrightarrow{g.} C. \longrightarrow 0$$

such that $A_n \to B_n \to C_n$ are split exact sequences for all n. Then $(\mathbf{Ch}(\mathcal{A}), \mathcal{E})$ is an exact category.

Lemma 1. *Consider a diagram*

$$\begin{array}{ccccc} & & & & E \\ & & & & \downarrow{\scriptstyle k} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & & & \\ D & & & & \end{array}$$

in an exact category with the second row a conflation. Then this diagram can be completed to a commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{f'} & B \oplus_C E & \xrightarrow{p_1} & E \\
\| & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle k} \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \iota_2} & & \| \\
D & \xrightarrow[\iota_1]{} & D \oplus^A B & \xrightarrow[g']{} & C
\end{array}$$

with all rows conflations.

Proof. Consider the two maps f and h. Since f is an inflation, (6) of definition 1 means the pushout $D \oplus^A B$ exists with ι_1 an inflation. Hence, there is a morphism $\eta : D \oplus^A B \to F$ such that $D \to D \oplus^A B \to F$ is a conflation. In particular, η is a cokernel of ι_1 so the pair (F, η) is unique up to isomorphism. Hence, we just need to show g' exists and that (C, g') is a cokernel of ι_1.

Consider the pair of maps $(g, 0)$ with $0 : D \to C$. Since $gf = 0 = 0h$ we know there is a unique map $g' : D \oplus^A B \to C$ such that $g = g'\iota_2$ and $0 = g'\iota_1$. Suppose $\varphi : D \oplus^A B \to W$ is a map for which $\varphi\iota_1 = 0$. Then $\varphi\iota_2 f = \varphi\iota_1 h = 0$ so there is a unique map $\alpha : C \to W$ such that $\varphi\iota_2 = \alpha g$. Hence, $(\varphi - \alpha g')\iota_1 = 0$ and $(\varphi - \alpha g')\iota_2 = \varphi\iota_2 - \alpha g'\iota_2 = 0$ and we find $\varphi - \alpha g' = 0$. Thus, φ factors uniquely through g' showing (C, g') is a cokernel of ι_1. The proof for the upper row is similar. □

Definition 2. If $(\mathcal{A}, \mathcal{E})$ and $(\mathcal{A}', \mathcal{E}')$ are exact categories, an *exact functor* $F : \mathcal{A} \to \mathcal{A}'$ is a functor which takes conflations in $\mathcal{A}$ to conflations in $\mathcal{A}'$. Exact functors are necessarily additive.

Lemma 2. *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be an exact sequence in an additive category A. Then for any object D, the sequences of abelian groups

$$0 \longrightarrow \mathrm{Hom}(C, D) \xrightarrow{g^*} \mathrm{Hom}(B, D) \xrightarrow{f^*} \mathrm{Hom}(A, D)$$

$$0 \longrightarrow \mathrm{Hom}(D, A) \xrightarrow[f_*]{} \mathrm{Hom}(D, B) \xrightarrow[g_*]{} \mathrm{Hom}(D, C)$$

are exact. Moreover, if there exists an $f' : B \to A$ ($g' : C \to B$) such that $f'f = \mathrm{id}_A$ ($gg' = \mathrm{id}_C$) then f^* (g_*) is surjective.

Proof. Since g is epi, g^* will be monic as $g^*(\varphi) = \varphi g$ can be zero only when $\varphi = 0$. Also, $f^* g^* = (gf)^* = 0$. Suppose $f^*(\varphi) = \varphi f = 0$ where $\varphi : B \to D$. Since (f, g) is an exact sequence we know g is a cokernel of f so there is a unique map $\varphi' : C \to D$ such that $\varphi = \varphi' g = g^*(\varphi')$. Hence, the kernel of f^* is contained in the image of g^*. If $f'f = \mathrm{id}_A$ then $f^*(f')^* = \mathrm{id}_{\mathrm{Hom}(A,D)}$ showing f^* is surjective. The proof for the second sequence is similar. □

Lemma 3. *Let $\mathcal{A}$ be an additive category with*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

an exact sequence. Then the following are equivalent

1. *There is a morphism $f' : B \to A$ such that $f'f = \mathrm{id}_A$.*
2. *There is a morphism $g' : C \to B$ such that $gg' = \mathrm{id}_C$.*
3. *There is an isomorphism $\varphi : B \to A \oplus C$ which makes the diagram*

(1.1)
$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \| & & \downarrow{\scriptstyle\varphi} & & \| \\ A & \xrightarrow[\iota_A]{} & A \oplus C & \xrightarrow[\pi_C]{} & C \end{array}$$

commute.

Proof. (1)⇒(3). If f' exists let $\varphi : B \to A \oplus C$ be the map $\varphi = \iota_A f' + \iota_C g$. This definition makes diagram 1.1 commute. For any object D, applying $\mathrm{Hom}(-, D)$ to diagram 1.1 gives the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(C, D) & \xrightarrow{g^*} & \mathrm{Hom}(B, D) & \xrightarrow{f^*} & \mathrm{Hom}(A, D) & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle\varphi^*} & & \| & & \\ 0 & \longrightarrow & \mathrm{Hom}(C, D) & \xrightarrow[\pi_C^*]{} & \mathrm{Hom}(A \oplus C, D) & \xrightarrow[\iota_A^*]{} & \mathrm{Hom}(A, D) & \longrightarrow & 0 \end{array}$$

which commutes and has exact rows. By the 5-lemma for abelian groups we find φ^* is an isomorphism. Therefore, φ induces a natural isomorphism $\varphi^* : \mathrm{Hom}(A \oplus C, -) \to \mathrm{Hom}(B, -)$. Hence, φ is an isomorphism by the Yoneda embedding theorem.

(2)⇒(3). Similar to (1)⇒(3).

(3)⇒(1)&(2). Just define $f' = \pi_A \varphi$ and $g' = \varphi^{-1}\iota_C$. □

Definition 3. Let $(\mathcal{A}, \mathcal{E})$ be an exact category.

1. An object $I \in \mathcal{A}$ is $\mathcal{E}$*-injective* (or *injective for short) if the functor*
$$\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to \mathbf{Ab}$$
is exact. Here the category **Ab** *of abelian groups has the canonical exact structure.*
2. *An object $P \in \mathcal{A}$ is $\mathcal{E}$-projective (or projective) if the functor*
$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$$
is exact.
3. *$\mathcal{A}$ has enough injectives if any object A fits into a conflation*
$$A \to I \to B$$
with I injective.
4. *$\mathcal{A}$ has enough projectives if any object A fits into a conflation*
$$B \to P \to A$$
with P projective.

Example 3. In the category of modules over a ring R with the usual exact sequences the projectives and injectives have their usual interpretations. Module categories have enough projectives and enough injectives.

Example 4. If we define the conflations in a module category to be the split short exact sequences then every object is projective and injective. Trivially then, module categories with split exact sequences as conflations have enough injectives and projectives.

Proposition 1. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Let I be an object of $\mathcal{A}$. Then the following are equivalent*

(1) *I is an injective object.*
(2) *For any diagram*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & \swarrow{\scriptstyle \exists h'} & & & \\ I & & & & \end{array}$$

with the top row a conflation, there is a morphism $h' : B \to I$ such that $h'f = h$.
(3) *Any conflation $I \to B \to C$ beginning with I splits, i.e, there is a map $h : B \to I$ such that $hf = \mathrm{id}_I$.*

Proof. (1)⇒(2). Applying $\mathrm{Hom}(-, I)$ to the conflation yields the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(C, I) \xrightarrow{g^*} \mathrm{Hom}(B, I) \xrightarrow{f^*} \mathrm{Hom}(A, I) \longrightarrow 0.$$

Since f^* is surjective and $h \in \mathrm{Hom}(A, I)$ there is an $h' \in \mathrm{Hom}(B, I)$ such that $h'f = f^*(h') = h$.

(2)⇒(3). Let

$$I \xrightarrow{f} B \xrightarrow{g} C$$

be a conflation. By (2), there is a map $h : B \to I$ such that $hf = \mathrm{id}_I$.

(3)⇒(1). Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a conflation. Let $h : A \to I$ be a morphism. By (6) of the definition of an exact category we can form the pushout diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} & & \| \\ I & \xrightarrow[f']{} & D & \xrightarrow[g']{} & C \end{array}$$

to get a conflation (f', g') beginning with I. Applying the functor $\mathrm{Hom}(-, I)$ we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(C,I) & \xrightarrow{g^*} & \mathrm{Hom}(B,I) & \xrightarrow{f^*} & \mathrm{Hom}(A,I) & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle k^*} & & \uparrow{\scriptstyle h^*} & & \\
0 & \longrightarrow & \mathrm{Hom}(C,I) & \xrightarrow[g'^*]{} & \mathrm{Hom}(D,I) & \xrightarrow[f'^*]{} & \mathrm{Hom}(I,I) & \longrightarrow & 0.
\end{array}$$

By assumption, there is a map $\pi : D \to I$ such that $\pi f' = \mathrm{id}_I$, i.e, $f'^*(\pi) = \mathrm{id}_I$. Using the commutativity of the diagram we get

$$f^* k^*(\pi) = h^* f'^*(\pi) = h^*(\mathrm{id}_I) = h$$

showing f^* is surjective. Hence, $\mathrm{Hom}(-, I)$ is an exact functor. □

Proposition 2. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category with P an object of $\mathcal{A}$. Then the following are equivalent:*

(1) *P is a projective object.*
(2) *For any diagram*

$$\begin{array}{ccccc}
 & & & & P \\
 & & & \swarrow^{\exists\, h'} & \downarrow{\scriptstyle h} \\
A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C
\end{array}$$

with the bottom row a conflation, there is a morphism $h' : P \to B$ such that $gh' = h$.
(3) *Any conflation $A \to B \to P$ ending with P splits.*

Definition 4. A *Frobenius category* is an exact category $\mathcal{A}$ such that:

(1) $\mathcal{A}$ has enough injectives.
(2) $\mathcal{A}$ has enough projectives.
(3) An object is injective if and only if it is projective.

Example 5. Let R be a quasi-Frobenius ring. Then the category of right (or left) modules with the canonical exact sructure is a Frobenius category by the Faith-Walker theorem.

Example 6. Let $(\mathbf{Ch}(\mathcal{A}), \mathcal{E})$ be the exact category of example 2. Then one can show that a complex is injective if and only if it is a split exact complex if and only if it is projective. One can use this fact to show that $\mathbf{Ch}(\mathcal{A})$ has enough injectives and projectives so is a Frobenius category.

2. Ideals and Quotient Categories

Definition 5. Let $\mathcal{A}$ be a pre-additive category and $\mathcal{I}$ a class of morphisms of $\mathcal{A}$. Denote $\mathcal{I}(A, B) = \mathcal{I} \cap \mathrm{Hom}_{\mathcal{A}}(A, B)$. A *two-sided ideal* (or ideal for short) $\mathcal{I}$ of $\mathcal{A}$ is a class of morphisms such that:

(1) For each pair of objects A and B in $\mathcal{A}$, $\mathcal{I}(A, B)$ is a subgroup of $\mathrm{Hom}_{\mathcal{A}}(A, B)$.
(2) If $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, $g \in \mathcal{I}(B, C)$ and $h \in \mathrm{Hom}_{\mathcal{A}}(C, D)$ then $hgf \in \mathcal{I}(A, D)$.

Equivalently, an ideal is a subfunctor of the bifunctor $\mathrm{Hom}_{\mathcal{A}}(-, -) : \mathcal{A}^{op} \times \mathcal{A} \to \mathbf{Ab}$ where the morphisms of the natural transformation are just inclusions.

If $\mathcal{A}$ is a preadditive category and $\mathcal{I}$ an ideal in $\mathcal{A}$ then we can define a new category $\mathcal{A}/\mathcal{I}$ as follows. The objects of $\mathcal{A}/\mathcal{I}$ are the same as the objects of $\mathcal{A}$. For any pair of objects A and B, the abelian group

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A,B) := \frac{\mathrm{Hom}_{\mathcal{A}}(A,B)}{\mathcal{I}(A,B)}$$

are the morphisms from A to B. Suppose $f - f' \in \mathcal{I}(A,B)$ and $g - g' \in \mathcal{I}(B,C)$. Then

$$gf - g'f' = gf - gf' + gf' - g'f' = g(f - f') + (g - g')f' \in \mathcal{I}(A,C)$$

showing $\overline{gf} = \overline{g'f'}$. Hence, $(\overline{f}, \overline{g}) \mapsto \overline{gf}$ is a well defined map

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(B,C) \times \mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A,B) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A,C).$$

This is how we define composition. The element $\overline{\mathrm{id}_A}$ is an identity for A and $\overline{0}_{(A,B)}$ is the zero object in $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A,B)$. Note, this composition is associative since it is associative in $\mathcal{A}$. By definition of the group structure on $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A,B)$ and composition we get

$$\overline{h}(\overline{g} + \overline{g'})\overline{f} = \overline{h}(\overline{g+g'})\overline{f} = \overline{h(g+g')f} = \overline{hgf + hg'f} = \overline{hgf} + \overline{hg'f} = \overline{h}\,\overline{g}\,\overline{f} + \overline{h}\,\overline{g'}\,\overline{f}$$

showing $\mathcal{A}/\mathcal{I}$ is preadditive. The functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ which is the identity on objects and which sends f to $\overline{f}$ is an additive functor which will be called the quotient functor.

If $\mathcal{A}$ is an additive category then $\mathcal{A}/\mathcal{I}$ is also additive for any ideal $\mathcal{I}$. If θ is a zero object of $\mathcal{A}$ then θ will also be a zero object in $\mathcal{A}/\mathcal{I}$ since $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(\theta, B)$ and $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(A, \theta)$ still consist of only one element. Let A and B be two objects of $\mathcal{A}$ with $A \oplus B$ the coproduct. Let ι_A, ι_B be the canonical inclusions and π_A, π_B the canonical projections. Since the quotient functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is additive we find

$$\begin{aligned} \overline{\pi_i}\,\overline{\iota_i} = \overline{\pi_i \iota_i} = \overline{\mathrm{id}}_i &\qquad \text{for} \quad i = A, B \\ \overline{\pi_i}\,\overline{\iota_j} = 0 &\qquad \text{for} \quad i \neq j \\ \overline{\iota_A}\,\overline{\pi_A} + \overline{\iota_B}\,\overline{\pi_B} = \overline{\mathrm{id}}_{A\oplus B} & \end{aligned}$$

showing $A \oplus B$ along with the maps $\overline{\iota_A}, \overline{\iota_B}$ are a coproduct of A and B in $\mathcal{A}/\mathcal{I}$. This proves the following proposition.

Proposition 3. *Let $\mathcal{A}$ be a (pre)additive category and $\mathcal{I}$ a two-sided ideal in $\mathcal{A}$. Then the quotient category $\mathcal{A}/\mathcal{I}$ is (pre)additive and the quotient functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is additive.*

The quotient category is a solution to a universal problem.

Theorem 1. *Let $G : \mathcal{A} \to \mathcal{B}$ be an additive functor between two (pre)additive categories and $\mathcal{I}$ an ideal in $\mathcal{A}$. If for any pair of objects A and A' in $\mathcal{A}$, $G(f) = 0$ for all $f \in \mathcal{I}(A, A')$, then there is a unique additive functor $H : \mathcal{A}/\mathcal{I} \to \mathcal{B}$ such that $G = HF$.*

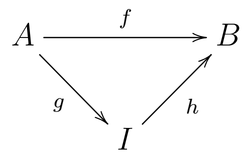
Proof. Since F is the identity on objects we have no choice but to define $H(A) = G(A)$ for all objects A. For any pair of objects (A, A'), $G(\mathcal{I}(A, A')) = \{0\}$ by assumption. Hence, there is a unique group homomorphism $H(A, A') : \mathrm{Hom}_{\mathcal{A}}(A, A')/\mathcal{I}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(G(A), G(A')) = \mathrm{Hom}_{\mathcal{B}}(H(A), H(A'))$ such that $G = HF$. H is given explicitly as $H(\overline{f}) = G(f)$ for all $\overline{f} \in \mathrm{Hom}_{\mathcal{A}}(A, A')/\mathcal{I}(A, A')$. Hence, $H(\overline{g}\,\overline{f}) = H(\overline{gf}) = G(gf) = G(g)G(f) = H(\overline{G})H(\overline{f})$. Also, $H(\overline{\mathrm{id}_A}) = G(\mathrm{id}_A) = \mathrm{id}_{G(A)} = \mathrm{id}_{H(A)}$ showing H is a functor. H is additive as

$$H(\overline{f} + \overline{g}) = H(\overline{f+g}) = G(f+g) = G(f) + G(g) = H(\overline{f}) + H(\overline{g}).$$

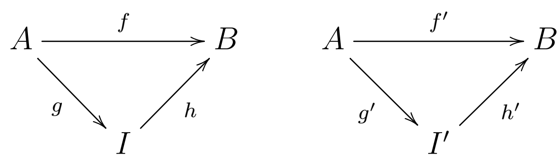
To see the uniqueness of H just note that any functor $H' : \mathcal{A}/\mathcal{I} \to \mathcal{B}$ with $G = H'F$ must be the identity on objects and must satisfy $H'(A, A')F(A, A') = G(A, A')$ for any pair (A, A'). This uniquely determines H' as being H defined above. □

3. The Stable Category of an Exact Category

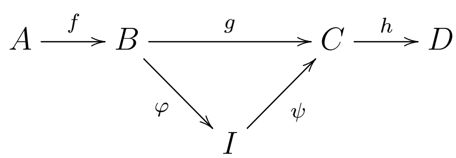
Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Let (A, B) be a pair of objects and define $\mathcal{I}(A, B) \subset \mathrm{Hom}_{\mathcal{A}}(A, B)$ to be the subset of all morphisms $f : A \to B$ which fit into a commutative diagram



with I an injective object. If $f = hg$ as in the diagram then $-f = (-h)g$. Also, zero morphisms factor through any object. If f and f' are in $\mathcal{I}(A, B)$ then we have two commutative diagrams



for injective objects I and I'. Let $\iota : I \to I \oplus I'$ and $\iota' : I' \to I \oplus I'$ be the canonical inclusions and $\pi : I \oplus I' \to I$ and $\pi' : I \oplus I' \to I'$ the canonical projections. $I \oplus I'$ is injective since I and I' are injective. Define $\varphi = \iota g + \iota' g' : A \to I \oplus I'$ and $\psi = h\pi + h'\pi' : I \oplus I' \to B$. Then $\psi\varphi = f + f'$ showing $f + f'$ factors through an injective object. Hence, $\mathcal{I}(A, B)$ is a subgroup of $\mathrm{Hom}_{\mathcal{A}}(A, B)$. If $f : A \to B$, $g \in \mathcal{I}(B, C)$ and $h : C \to D$ then the diagram



shows $hgf = h\psi\varphi f \in \mathcal{I}(A, D)$. Hence, $\mathcal{I}$ is an ideal in $\mathcal{A}$. Similarly, there is the ideal $\mathcal{P}$ consisting of all morphisms which factor through a projective object.

Definition 6. Let $\mathcal{A}$ be an exact category. The *injectively stable category*, $\underline{\mathcal{A}}$, is the quotient $\mathcal{A}/\mathcal{I}$ where $\mathcal{I}$ is defined as the class of all morphisms which factor through an injective objective. The *projectively stable category*, $\overline{\mathcal{A}}$, is the quotient $\mathcal{A}/\mathcal{P}$ with $\mathcal{P}$ the ideal of all morphisms which factor through a projective object.

The set of morphisms between two objects A and B in $\underline{\mathcal{A}}$ will be denoted $\underline{\mathrm{Hom}}(A, B)$. The residue class of a morphism $\varphi : A \to B$ will be denoted $\underline{\varphi}$. Similarly, $\overline{\mathrm{Hom}}(A, B)$ will denote the morphisms in $\overline{\mathcal{A}}$ and a morphism will be denoted by $\overline{\varphi}$.

Lemma 4. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category with $\mathcal{I}$ ($\mathcal{P}$) as above. Let A and B be two objects of $\mathcal{A}$. If there are injectives I and I' (projectives P, P') such that $A \oplus I \cong B \oplus I'$ ($A \oplus P \cong B \oplus P'$) then $A \cong B$ in $\underline{\mathcal{A}}$ ($\overline{\mathcal{A}}$).*

Proof. Any map from $A \oplus I \to B \oplus I'$ can be written uniquely as a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

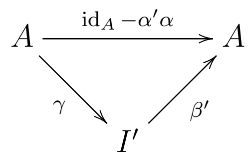
where $\alpha : A \to B$, $\beta : I \to B$, $\gamma : A \to I'$ and $\delta : I \to I'$. Similarly, there is a matrix

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

for any map from $B \oplus I' \to A \oplus I$. An isomorphism of the two direct sums gives matrices above such that

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha'\alpha + \beta'\gamma & \alpha'\beta + \beta'\delta \\ \gamma'\alpha + \delta'\gamma & \gamma'\beta + \delta'\delta \end{pmatrix} = \begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_I \end{pmatrix}$$

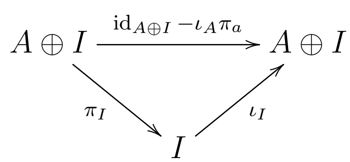
Hence, $\mathrm{id}_A - \alpha'\alpha = \beta'\gamma$ which means



commutes. Therefore, $\underline{\mathrm{id}_A} = \underline{\alpha'\alpha}$. Similarly, $\underline{\alpha\alpha'} = \underline{\mathrm{id}_B}$ showing A is isomorphic to B in $\underline{\mathcal{A}}$. □

Lemma 5. *If $(\mathcal{A}, \mathcal{E})$ is an exact category with A an object and I (P) an injective object (projective object). Then A is isomorphic to $A \oplus I$ $(A \oplus P)$ in $\underline{\mathcal{A}}$ $(\overline{\mathcal{A}})$.*

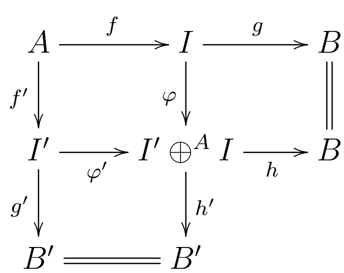
Proof. This is a corollary to the previous lemma but the particular isomorphism in this case is important. Since $\iota_A \pi_A + \iota_I \pi_I = \mathrm{id}_{A \oplus I}$ we find



commutes. Hence, $\underline{\iota_A} : A \to A \oplus I$ is an isomorphism in $\underline{\mathcal{A}}$ with inverse $\underline{\pi_A}$. □

Lemma 6. Schanuel's lemma. *Let $\mathcal{A}$ be an exact category. Given two conflations $A \to I \to B$ and $A \to I' \to B$ with I and I' injective, $B \oplus I' \cong B' \oplus I$. In particular, B and B' are isomorphic in the injectively stable category.*

Proof. Given the two conflations we can construct the commutative diagram



in which the middle row and column are conflations. Since I and I' are injective these conflations split showing $B' \oplus I \cong I' \oplus^A I \cong B \oplus I'$. □

Remark. There is a similar statement for conflations $B \to P \to A$ and $B' \to P' \to A$ with P, P' projective.

It will be useful to have a better understanding of the isomorphism $B \cong B'$ in Schanuel's lemma. Since

$$I \xrightarrow{\varphi} I' \oplus^A I \xrightarrow{h'} B'$$

is a conflation with I injective we know there is a map $\psi : I' \oplus^A I \to I$ such that $\psi\varphi = \mathrm{id}_I$. This gives the isomorphism $\alpha = \iota_I \psi + \iota_{B'} h'$. Define $k' = \alpha^{-1}\iota_{B'}$. Then $h'k' = \mathrm{id}_{B'}$, $k'h' + \varphi\psi = \mathrm{id}_{I' \oplus^A I}$ and we can write $\alpha^{-1} = \varphi\pi_{I'} + k'\pi_{B'}$. Similarly, for the conflation

$$I' \xrightarrow{\varphi'} I' \oplus^A I \xrightarrow{h} B$$

we have maps $\psi' : I' \oplus^A I \to I'$ and $k : B \to I' \oplus^A I$ such that $\beta = \iota_{I'}\psi' + \iota_B h : I' \oplus^A I \cong I' \oplus B$, $\beta^{-1} = \varphi'\pi_{I'} + k'\pi_B$ and $\varphi'\psi' + kh = \mathrm{id}_{I' \oplus^A I}$. Using these maps we get isomorphisms $\alpha\beta^{-1} : B \oplus I' \to B' \oplus I$ and $\beta\alpha^{-1} : B' \oplus I \to B \oplus I'$ which when written as matrices have the form

$$\alpha\beta^{-1} = \begin{pmatrix} h'k & h'\varphi' \\ \psi k & \psi\varphi' \end{pmatrix}, \qquad \beta\alpha^{-1} = \begin{pmatrix} hk' & h\varphi \\ \psi' k' & \psi'\varphi \end{pmatrix}.$$

Hence, by lemma 4, the map $\underline{h'k} : B \to B'$ is an isomorphism in $\underline{\mathcal{A}}$ with inverse $\underline{hk'}$. Let $\gamma = \psi'\varphi : I \to I'$ and $\gamma' = -h'k$. Then

$$\gamma f = \psi'\varphi f = \psi'\varphi' f' = \mathrm{id}_{I'} f' = f'$$

and

$$g'\gamma = g'\psi'\varphi = h'\varphi'\psi'\varphi = h'(\varphi'\psi' + kh)\varphi - h'kh\varphi = h'\varphi - h'kg = \gamma' g.$$

Therefore, we have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & I & \xrightarrow{g} & B \\ \| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma'} \\ A & \xrightarrow[f']{} & I' & \xrightarrow[g']{} & B' \end{array}$$

in which $\underline{\gamma'} : B \to B'$ is an isomorphism in $\underline{\mathcal{A}}$. By similar reasoning, given two conflations $B \to P \to A$ and $B' \to P' \to A$ with P and P' projective, there is a commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & P & \longrightarrow & A \\ {\scriptstyle \gamma'}\downarrow & & \downarrow{\scriptstyle \gamma} & & \| \\ B' & \longrightarrow & P' & \longrightarrow & A \end{array}$$

with $\overline{\gamma'}$ an isomorphism in $\overline{\mathcal{A}}$.

Let $(\mathcal{A}, \mathcal{E})$ be an exact category with enough injectives. Then we can define an additive functor $T : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$. In the case of a Frobenius category, this T will be an auto-equivalence.

Suppose we have two conflations

$$A \xrightarrow{f} I \xrightarrow{g} A'$$

$$B \xrightarrow[f']{} I' \xrightarrow[g']{} B'$$

with I and I' injective. Let $\varphi : A \to B$ be a morphism in $\mathcal{A}$. Since $f'\varphi : A \to I'$ is a morphism into an injective, proposition 1 gives the existence of a map $\varphi' : I \to I'$ such that the first square in the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & I & \xrightarrow{g} & A' \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi''} \\ B & \xrightarrow[f']{} & I' & \xrightarrow[g']{} & B' \end{array}$$

commutes. Then we get a unique $\varphi'' : A' \to B'$ filling in the second dotted line making the right square commute as $g'\varphi' f = g' f' \varphi = 0$.

Suppose we have $\psi' : I \to I'$ and $\psi'' : A' \to B'$ such that $\psi'' g = g'\psi'$ and $\psi' f = f'\varphi$. Then $(\varphi' - \psi')f = f'(\varphi - \varphi) = 0$ which implies the existence of a map $s : A' \to I'$ such that $\varphi' - \psi' = sg$. Then

$$(\varphi'' - \psi'')g = g'(\varphi' - \psi') = g'sg$$

which implies $\varphi'' - \psi'' = g's$ since g is epi. Hence, $\varphi'' - \psi''$ factors through I' showing $\underline{\varphi''} = \underline{\psi''}$. Hence, the map $\varphi : A \to B$ induces a map $\varphi'' : A' \to B'$ which is unique in $\underline{\mathcal{A}}$.

Similarly, If $\mathcal{A}$ is an exact category with enough projectives and we have two conflations

$$A' \xrightarrow{f} P \xrightarrow{g} A$$

$$B' \xrightarrow[f']{} P' \xrightarrow[g']{} B$$

with P and P' projective, then any map $\varphi : A \to B$ induces a map $\varphi'' : A' \to B'$ which is unique in $\overline{\mathcal{A}}$.

Now we can define $T : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$. We will construct T in stages. First we define $T : \mathcal{A} \to \underline{\mathcal{A}}$, then show T sends morphisms in $\mathcal{I}$ to zero morphisms. This will give us the functor $T : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$. For every object A in $\mathcal{A}$, fix a conflation

$$A \xrightarrow{u(A)} I(A) \xrightarrow{\pi(A)} TA \tag{3.1}$$

with $I(A)$ injective. For each object A, define $T(A) = TA$ where TA comes from 3.1. Given a map $\varphi : A \to B$, define $T(\varphi) = \underline{\varphi''}$ where $\varphi'' : TA \to TB$ is constructed as in the previous paragraph. If $\varphi = \mathrm{id}_A$ then we may take $\varphi' = \mathrm{id}_{I(A)}$ which gives $\varphi'' = \mathrm{id}_{TA}$. This gives $T(\mathrm{id}_A) = \underline{\mathrm{id}_{TA}}$. Similarly, given $\psi : B \to C$ we can take $\psi''\varphi''$ for $(\psi\varphi)'' : TA \to TC$ showing $T(\psi\varphi) = \underline{\psi''\varphi''} = \underline{\psi''}\ \underline{\varphi''} = T(\psi)T(\varphi)$. Moreover, given $\varphi, \psi : A \to B$ we can take $\varphi'' + \psi''$ for $(\varphi + \psi)''$ showing $T(\varphi + \psi) = \underline{(\varphi + \psi)''} = \underline{\varphi'' + \psi''} = \underline{\varphi''} + \underline{\psi''} = T(\varphi) + T(\psi)$. Hence, T

is an additive functor from $\mathcal{A}$ to $\underline{\mathcal{A}}$.

If $f \in \mathcal{I}(A, B)$ then we can write $f = hg$ where $g : A \to I$ and $h : I \to B$ for some injective object I. Therefore, $f'' = h''g''$ using the same notation as before where $g'' : TA \to TI$ and $h'' : TI \to TB$. Since $I \to I(I) \to TI$ is a conflation starting with an injective it splits. Hence, $I(I) \cong I \oplus T(I)$ showing $T(I)$ is injective. Therefore, $f'' = h''g''$ factors through a projective showing $\underline{f''} = 0$. Hence, $T(f) = 0$. This holds for all morphisms in $\mathcal{I}$ so by the universal property of $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{I}$, we know T induces a unique additive functor, which will also be denoted as T, from $\underline{\mathcal{A}} \to \underline{\mathcal{A}}$.

One can, in a similar fashion, construct an additive functor $S : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ whenever $\mathcal{A}$ has enough projectives. This is done by choosing for each object $\mathcal{A}$, a conflation

$$SA \xrightarrow{\iota(A)} P(A) \xrightarrow{p(A)} A$$

with $P(A)$ projective. Then S is defined on objects via $S(A) = SA$ and on morphisms $f : A \to B$ as $T(f) = \overline{f''}$ where f'' is constructed such that

$$\begin{array}{ccccc} SA & \xrightarrow{\iota(A)} & P(A) & \xrightarrow{p(A)} & A \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ SB & \xrightarrow[\iota(B)]{} & P(B) & \xrightarrow[p(B)]{} & B \end{array}$$

commutes. Again, $\overline{f''}$ does not depend on the choice of f'.

4. A triangulation of the Stable Category of a Frobenius Category.

Let $(\mathcal{A}, \mathcal{E})$ be a Frobenius category. Recall this means $\mathcal{A}$ has enough injectives, enough projectives and an object is injective if and only if it is projective. In this case, the injectively stable and projectively stable categories are the same, $\underline{\mathcal{A}} = \overline{\mathcal{A}}$. We will stick with the underline notation.

Theorem 2. *Let $\mathcal{A}$ be a Frobenius category. The functor $T : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ constructed in the previous section is an auto-equivalence with S being a quasi-inverse.*

Proof. Let A be an object of $\underline{\mathcal{A}}$. $S(A) = SA$ is the object of a chosen conflation

$$SA \xrightarrow{\iota(A)} P(A) \xrightarrow{p(A)} A.$$

We use the conflation

$$SA \xrightarrow{u(SA)} I(SA) \xrightarrow{\pi(SA)} TSA$$

to get $TS(A)$. By Schanuel's lemma and the discussion afterwards, and the fact that projectives and injectives coincide we can construct the following commutative diagram

$$\begin{array}{ccccc} SA & \xrightarrow{u(SA)} & I(SA) & \xrightarrow{\pi(SA)} & TSA \\ \Vert & & \downarrow & & \downarrow{\scriptstyle \eta_A} \\ SA & \xrightarrow[\iota(A)]{} & P(A) & \xrightarrow[p(SA)]{} & A \end{array}$$

such that $\eta_A : TSA \to A$ is an isomorphism in $\underline{\mathcal{A}}$.

Suppose we have a map $f : A \to B$. Then we define $S(f)$ to be $\overline{f''} = \underline{f''}$ where f'' is constructed to make

$$\begin{array}{ccccc} SA & \xrightarrow{\iota(A)} & P(A) & \xrightarrow{p(A)} & A \\ {\scriptstyle f''}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ SB & \xrightarrow[\iota(B)]{} & P(B) & \xrightarrow[p(B)]{} & B \end{array}$$

commute. From here we can construct two commutative diagrams, being loose with the identification of maps in $\underline{\mathcal{A}}$ and a representative in $\mathcal{A}$,

$$\begin{array}{ccccc} SA & \xrightarrow{u(SA)} & I(SA) & \xrightarrow{\pi(SA)} & TSA \\ \| & & \downarrow & & \downarrow{\scriptstyle \eta_A} \\ SA & \xrightarrow{\iota(A)} & P(A) & \xrightarrow{p(A)} & A \\ {\scriptstyle S(f)}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ SB & \xrightarrow[\iota(B)]{} & P(B) & \xrightarrow[p(B)]{} & B \end{array} \qquad \begin{array}{ccccc} SA & \xrightarrow{u(SA)} & I(SA) & \xrightarrow{\pi(SA)} & TSA \\ {\scriptstyle S(f)}\downarrow & & \downarrow & & \downarrow{\scriptstyle TS(f)} \\ SB & \xrightarrow{u(SB)} & I(SB) & \xrightarrow{\pi(SB)} & TSB \\ \| & & \downarrow & & \downarrow{\scriptstyle \eta_B} \\ SB & \xrightarrow[\iota(B)]{} & P(B) & \xrightarrow[p(B)]{} & B. \end{array}$$

From the two diagrams we find $\underline{f}\eta_A$ and $TS(f)\eta_B$ are maps which are induced by $S(f) : SA \to SB$. Hence,

$$\underline{f}\eta_A = TS(f)\eta_B$$

showing $\eta : TS \to \mathrm{Id}_{\underline{\mathcal{A}}}$ is a natural isomorphism. Similarly, we can construct a natural isomorphism $\nu : ST \to \mathrm{Id}_{\underline{\mathcal{A}}}$ showing T is an equivalence with S a quasi-inverse. □

Lets recall the definition of a triangulated category. Suppose $\mathcal{A}$ is an additive category with an autoequivalence T. A triangle in $\mathcal{A}$ is a sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA.$$

A (iso)morphism of triangles is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle T(\varphi)} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA' \end{array}$$

such that φ, ψ and θ are (iso)morphisms.

Definition 7. A triangulated category is an additive category with an auto-equivalence T along with a class of triangles $\mathcal{T}$, called exact triangles, such that the following hold:

(TR1) Every triangle isomorphic to an exact triangle is exact. Every morphism $f : A \to B$ can be embedded into an exact triangle as

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow TA.$$

For any object A, the triangle

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow TA$$

is exact.

(TR2) If

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

is an exact triangle then

$$B \xrightarrow{g} C \xrightarrow{h} TA \xrightarrow{-Tf} TB.$$

is an exact triangle.

(TR3) Given a diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \vdots & & \downarrow{\scriptstyle T(\varphi)} \\
A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & TA'
\end{array}$$

with the rows exact triangles and $\psi f = f'\varphi$, there exists a map θ filling in the dotted arrow making the diagram commute.

(TR4) **The Octohedral Axiom** Let $f : A \to B$ and $g : B \to C$ be two morphisms. If we have exact triangles

$$A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{f''} TA$$

$$B \xrightarrow{g} C \xrightarrow{g'} E \xrightarrow{g''} TB$$

$$A \xrightarrow{h=gf} C \xrightarrow{h'} F \xrightarrow{h''} TA$$

then there is an exact triangle

$$D \xrightarrow{j} F \xrightarrow{j'} E \xrightarrow{j''} TD$$

such that the diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{f'} & D & \xrightarrow{f''} & TA \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle j} & & \| \\
A & \xrightarrow[h=gf]{} & C & \xrightarrow{h'} & F & \xrightarrow{h''} & TA \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle T(f)} \\
 & & E & = & E & \xrightarrow[g'']{} & TB \\
 & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle j''} & & \\
 & & TB & \xrightarrow[T(f')]{} & TD & &
\end{array}$$

commutes.

Remark. J.P. May shows in [4] that TR3 follows from TR1, TR2 and TR4.

Recall that a *pre-triangulated* category is a category satisfying all conditions in definition 7 except TR4. Also, the triangulated 5-lemma holds in a pre-triangulated category.

We will define a class of triangles in $\underline{\mathcal{A}}$ and show it is a triangulation of $\underline{\mathcal{A}}$. Let $f : A \to B$ be a morphism, and $A \to I \to A'$ a conflation with I injective. Let $B \oplus^A I$ be a pushout and form the diagram

$$\begin{array}{ccccc} A & \xrightarrow{u} & I & \xrightarrow{\pi} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} & & \| \\ B & \xrightarrow[g]{} & B \oplus^A I & \xrightarrow[h']{} & A' \end{array}$$

in which the bottom row is a conflation. Let $\nu : A' \to TA$ be a morphism such that $\underline{\nu}$ is an isomorphism. Then, letting $C = B \oplus^A I$ and $h = \nu h'$, we get a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

which will be called *standard*. Define $\mathcal{T}$ to be the class of all triangles which are isomorphic to a standard triangle. $\mathcal{T}$ will be the class of exact triangles.

Theorem 3. *The category $\underline{\mathcal{A}}$ with the auto-equivalence T and the class $\mathcal{T}$ is a pre-triangulated category.*

Proof. (TR1). Every triangle isomorphic to an exact triangle is, by composition, isomorphic to a standard triangle. Every morphism $f : A \to B$ can be placed into an exact triangle since they can always be placed into a standard triangle. From the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\underline{\mathrm{id}}} & A & \xrightarrow{\underline{u(A)}} & I(A) & \xrightarrow{\underline{\pi(A)}} & TA \\ \downarrow{\scriptstyle \underline{\mathrm{id}}} & & \downarrow{\scriptstyle \underline{\mathrm{id}}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \underline{\mathrm{id}}} \\ A & \xrightarrow[\underline{\mathrm{id}}]{} & A & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & TA \end{array}$$

and the fact that $I(A) \cong 0$ in $\underline{\mathcal{A}}$ we get the bottom sequence is an exact triangle.

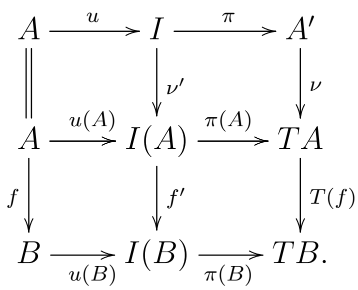
(TR2) It is enough to consider standard triangles. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

be a standard triangle constructed from a commutative diagram of the form

$$\begin{array}{ccccccc} A & \xrightarrow{u} & I & \xrightarrow{\pi} & A' & & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} & & \| & & \\ B & \xrightarrow[g]{} & C & \xrightarrow[h']{} & A' & \xrightarrow[\nu]{} & TA \end{array}$$

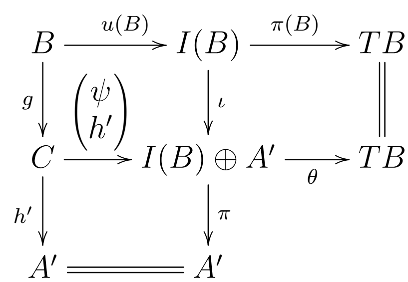
Here, ν and $T(f)$ will be constructed from the following commutative diagram



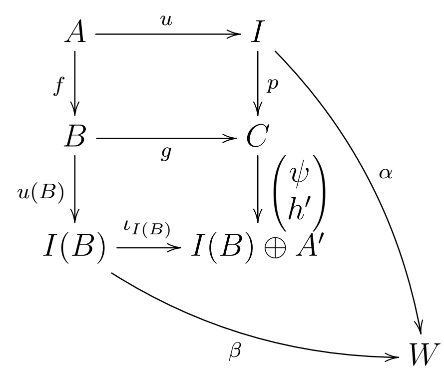
Since $f'\nu'u = u(B)f$ and C is a pushout we get a unique $\psi : C \to I(B)$ such that $\psi g = u(B)$ and $\psi p = f'\nu'$. Since

$$\begin{aligned}(T(f)\nu h' - \pi(B)\psi)g &= T(f)\nu h'g - \pi(B)\psi g = 0 - \pi(B)u(B) = 0 \text{ and} \\ (T(f)\nu h' - \pi(B)\psi)p &= T(f)\nu\pi(A) - \pi(B)\varphi = 0\end{aligned}$$

we know $T(f)\nu h' = \pi(B)\psi$. Therefore, we can construct the diagram



in which the first two columns and the top row are conflations. Here, $\theta = (\pi(B), -T(f)\nu)$. Suppose we have maps $\alpha : I \to W$ and $\beta : I(B) \to W$ as in the diagram



such that $\alpha u = \beta u(B)f$. Then we get

$$\alpha u = \beta\psi g f = \beta\psi p u = \beta f'\nu' u.$$

Hence, there is a unique map $\gamma : A' \to W$ such that $\alpha - \beta f'\nu' = \gamma\pi$. Define $\delta : I(B) \oplus A' \to W$ as $\delta := \beta\pi_{I(B)} + \gamma\pi_{A'}$. Then $\delta\iota_{I(B)} = \beta$ and

$$\delta\begin{pmatrix}\psi \\ h'\end{pmatrix} p = (\beta\psi + \gamma h')p = \beta\psi p + \gamma h'p = \beta f'\nu' + \gamma\pi = \alpha.$$

The uniqueness of δ comes from the uniqueness of γ. Hence, the rectangle is a pushout. However, the upper square in the rectangle is a pushout so by the pushout lemma, the bottom square of the rectangle is a pushout. Since $u(B)$ is an inflation we get $\begin{pmatrix}\psi\\ h'\end{pmatrix}$ is an inflation. Notice

$$\theta\begin{pmatrix}\psi\\ h'\end{pmatrix} = (\pi(B), -T(f)\nu)\begin{pmatrix}\psi\\ h'\end{pmatrix} = \pi(B)\psi - T(f)\nu h' = 0.$$

Suppose we have a map $(a, -b) : I(B) \oplus A' \to W$ such that

$$(a, b)\begin{pmatrix}\psi\\ h'\end{pmatrix} = a\psi - bh' = 0.$$

Using the fact that $h'g = 0$ and $u(B) = \psi g$ we find

$$au(B) = a\psi g = a\psi g - bh'g = (a\psi - bh')g = 0.$$

Therefore, there is a unique $a' : TB \to W$ such that $a = a'\pi(B)$. Also,

$$bh' = a\psi = a'\pi(B)\psi = a'T(f)\nu h'$$

implies $b = a'T(f)$ as h' is epi. Hence, $(a, -b) = (a'\pi(B), -a'T(f)\nu) = a'(\pi(B), -T(f)\nu) = a'\theta$ showing θ is a cokernel of $\begin{pmatrix}\psi\\ h'\end{pmatrix}$. Hence, the first row in the diagram below is a standard triangle:

$$\begin{array}{ccccccc}
B & \xrightarrow{g} & C & \xrightarrow{\begin{pmatrix}\psi\\ h'\end{pmatrix}} & I(B)\oplus A' & \xrightarrow{\theta} & TB \\
\| & & \| & & \downarrow{\scriptstyle \pi_{A'}} & & \| \\
B & \xrightarrow{g} & C & \xrightarrow{h'} & A' & \xrightarrow{-T(f)\nu} & TB \\
\| & & \| & & \downarrow{\scriptstyle \nu} & & \| \\
B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & TA. & \xrightarrow[-T(f)]{} & TB.
\end{array}$$

All squares except the top right corner commute. However,

$$\theta - T(f)\nu = (\pi(B), 0)$$

shows $\theta - T(f)\nu$ factors through an injective showing the whole diagram commutes in $\underline{\mathcal{A}}$. Also, $\underline{\pi_{TA}}$ and $\underline{\nu}$ are isomorphisms showing the lower triangle is isomorphic to the upper triangle in $\underline{\mathcal{A}}$. Hence, the lower triangle is an exact triangle.

(TR3) Again, it is enough to consider standard triangles. Suppose we have two standard triangles

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & I & \xrightarrow{\pi} & A' & & \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha} & & \| & & \\
B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & A' & \xrightarrow[\nu]{} & TA
\end{array}$$

and

$$\begin{array}{ccccccc}
X & \xrightarrow{u'} & J & \xrightarrow{\pi'} & X' & & \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \alpha'} & & \| & & \\
Y & \xrightarrow[g']{} & Z & \xrightarrow[h']{} & X' & \xrightarrow[\mu]{} & TX
\end{array}$$

and the following commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{\underline{f}} & B & \xrightarrow{\underline{g}} & C & \xrightarrow{\underline{h}} & TA \\
{\scriptstyle \underline{\varphi}}\downarrow & & \downarrow{\scriptstyle \underline{\psi}} & & & & \downarrow{\scriptstyle T(\varphi)} \\
X & \xrightarrow[\underline{f'}]{} & Y & \xrightarrow[\underline{g'}]{} & Z & \xrightarrow[\underline{h'}]{} & TX.
\end{array} \tag{4.1}$$

Since $\underline{\nu}$ is an isomorphism we can find a ω such that $\underline{\omega\nu} = \mathrm{id}_{A'}$ and $\underline{\nu\omega} = \mathrm{id}_{TA}$. Hence, we can construct the commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{u(A)} & I(A) & \xrightarrow{\pi(A)} & TA \\
\| & & \downarrow{\scriptstyle \omega'} & & \downarrow{\scriptstyle \omega} \\
A & \xrightarrow{u} & I & \xrightarrow{\pi} & A' \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi''} \\
X & \xrightarrow[u']{} & J & \xrightarrow[\pi']{} & X' \\
\| & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu} \\
X & \xrightarrow[u(X)]{} & I(X) & \xrightarrow[\pi(X)]{} & TX
\end{array}$$

and get $T(\varphi) = \underline{\mu\varphi''\omega}$. As $\underline{\psi f} = \underline{f'\varphi}$ there is an injective I' such that $\psi f - f'\varphi$ factors through I:

$$\begin{array}{ccc}
A & \xrightarrow{\psi f - f'\varphi} & Y \\
& {\scriptstyle \epsilon}\searrow \quad \nearrow{\scriptstyle \beta} & \\
& I'. &
\end{array}$$

Since ϵ factors through u we may as well assume $I' = I$ and $\epsilon = u$. Thus,

$$\psi f = f'\varphi + \beta u.$$

Consider the two morphisms $g'\psi : B \to Z$ and $\alpha'\varphi' + g'\beta : I \to Z$. Since C is a pushout and

$$g'\psi f = g'f'\varphi + g'\beta u = \alpha'u'\varphi + g'\beta u = \alpha'\varphi' u + g'\beta u = (\alpha'\varphi' + g'\beta)u,$$

we get a unique map $\theta : C \to Z$ such that $\theta g = g'\psi$ and $\theta\alpha = \alpha'\varphi' + g'\beta$. Using θ,

$$(h'\theta - \varphi'' h)\alpha = h'\alpha'\varphi' + h'g'\beta - \varphi''\pi(A) = \pi(A')\varphi' - \varphi''\pi(A) = 0$$

and

$$(h'\theta - \varphi'' h)g = h'\theta g - \varphi'' hg = h'g'\psi - \varphi'' hg = 0 - 0 = 0$$

showing $h'\theta = \varphi'' h$ using that C is a pushout. Hence, $\theta : C \to Z$ is a map for which

$$
\begin{array}{ccccccc}
A & \xrightarrow{\underline{f}} & B & \xrightarrow{\underline{g}} & C & \xrightarrow{\underline{h}} & A' \\
{\scriptstyle \underline{\varphi}}\downarrow & & \downarrow{\scriptstyle \underline{\psi}} & & \downarrow{\scriptstyle \underline{\theta}} & & \downarrow{\scriptstyle \underline{\varphi''}} \\
X & \xrightarrow[\underline{f'}]{} & Y & \xrightarrow[\underline{g'}]{} & Z & \xrightarrow[\underline{h'}]{} & X'.
\end{array}
$$

commutes. Hence, the larger diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{\underline{f}} & B & \xrightarrow{\underline{g}} & C & \xrightarrow{\underline{\nu h}} & TA \\
\| & & \| & & \| & & \downarrow{\scriptstyle \underline{\omega}} \\
A & \xrightarrow{\underline{f}} & B & \xrightarrow{\underline{g}} & C & \xrightarrow{\underline{h}} & A' \\
{\scriptstyle \underline{\varphi}}\downarrow & & \downarrow{\scriptstyle \underline{\psi}} & & \downarrow{\scriptstyle \underline{\theta}} & & \downarrow{\scriptstyle \underline{\varphi''}} \\
X & \xrightarrow[\underline{f'}]{} & Y & \xrightarrow[\underline{g'}]{} & Z & \xrightarrow{\underline{h'}} & X' \\
\| & & \| & & \| & & \downarrow{\scriptstyle \underline{\mu}} \\
X & \xrightarrow[\underline{f'}]{} & Y & \xrightarrow[\underline{g'}]{} & Z & \xrightarrow[\underline{\mu h'}]{} & TX
\end{array}
$$

commutes showing $\underline{\theta} : C \to Z$ fills in diagram 4.1 making it commute. ☐

Lemma 7. *Consider a morphism $f : A \to B$. Then any two standard triangles constructed from the pushout diagrams*

$$
\begin{array}{ccccc}
A & \xrightarrow{u} & I & \xrightarrow{\pi} & A' \\
{\scriptstyle f}\downarrow & & \downarrow & & \| \\
B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & A'
\end{array}
\qquad\qquad
\begin{array}{ccccc}
A & \xrightarrow{u'} & I' & \xrightarrow{\pi'} & A'' \\
{\scriptstyle f}\downarrow & & \downarrow & & \| \\
B & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & A''
\end{array}
$$

are isomorphic.

Proof. This follows from the commutative diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{\underline{f}} & B & \xrightarrow{\underline{g}} & C & \xrightarrow{\underline{\nu h}} & TA \\
\| & & \| & & & & \| \\
A & \xrightarrow[\underline{f}]{} & B & \xrightarrow[\underline{g'}]{} & C' & \xrightarrow[\underline{\nu' h'}]{} & TA,
\end{array}
$$

TR3 and the triangulated 5-lemma. ☐

Theorem 4. *The pre-triangulated category $\underline{\mathcal{A}}$ is triangulated.*

Proof. We just need to show the octohedral axiom holds. Suppose we have three exact triangles

$$A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{f''} TA$$

$$B \xrightarrow{g} C \xrightarrow{g'} E \xrightarrow{g''} TB$$

$$A \xrightarrow{h=gf} C \xrightarrow{h'} F \xrightarrow{h''} TA.$$

We can assume each triangle is standard. Actually, by the previous lemma we can assume the first and third triangles are constructed as

$$\begin{array}{ccccc} A & \xrightarrow{u(A)} & I(A) & \xrightarrow{\pi(A)} & TA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha} & & \| \\ B & \xrightarrow[f']{} & D & \xrightarrow[f'']{} & TA \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{u(A)} & I(A) & \xrightarrow{\pi(A)} & TA \\ {\scriptstyle h=gf}\downarrow & & \downarrow{\scriptstyle \gamma} & & \| \\ C & \xrightarrow[h']{} & F & \xrightarrow[h'']{} & TA \end{array}$$

and the second triangle is constructed as

$$\begin{array}{ccccccc} B & \xrightarrow{u(D)f'} & I(D) & \xrightarrow{\pi} & B' & & \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \beta} & & \| & & \\ C & \xrightarrow[g']{} & E & \xrightarrow[g'']{} & B' & \xrightarrow[\nu]{} & TB \end{array}$$

Note, as $u(D)$ and f' are inflations, their composition is an inflation which gives the conflation $B \to I(D) \to B'$. As D is a pushout and

$$(h'g)f = h'h = \gamma u(A)$$

there is a unique $j : D \to F$ such that $jf' = h'g$ and $j\alpha = \gamma$. Similarly,

$$g'h = g'gf = \beta u(D)f'f = \beta u(D)\alpha u(A)$$

implies there is a unique $j' : F \to E$ such that $j'h' = g'$ and $j'\gamma = \beta u(D)\alpha$. As

$$\begin{aligned} (j'j - \beta u(D))\alpha &= j'\gamma - \beta u(D)\alpha = \beta u(D)\alpha - \beta u(D)\alpha = 0 \text{ and} \\ (j'j - \beta u(D))f' &= j'h'g - \beta u(D)f' = g'g - g'g = 0 \end{aligned}$$

we get $j'j = \beta u(D)$ again using the fact that D is a pushout. Hence, we can construct the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{u(A)} & I(A) & & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha} & & \\ B & \xrightarrow{f'} & D & \xrightarrow{u(D)} & I(D) \\ {\scriptstyle g}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow[h']{} & F & \xrightarrow[j']{} & E. \end{array}$$

The top left square is a pushout by construction and since $gf = h$ and $j\alpha = \gamma$ we see the tall left rectangle is also a pushout. Hence, by the pushout lemma, the bottom left square is a pushout. However, $j'h' = g'$ shows the flat rectangle is a pushout by construction so we again use the pushout lemma to conclude the bottom right square is a pushout. We can use the bottom right square to construct the following standard triangle

$$\begin{array}{ccccc}
D & \xrightarrow{u(D)} & I(D) & \xrightarrow{\pi(D)} & TD \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \beta} & & \| \\
F & \xrightarrow[j']{} & E & \xrightarrow[j'']{} & TD.
\end{array}$$

The map $\underline{\nu} : B' \to TB$ was chosen to be an isomorphism, let $\underline{\mu}$ be an inverse. We will use the diagrams

$$\begin{array}{ccccc}
A & \xrightarrow{u(A)} & I(A) & \xrightarrow{\pi(A)} & TA \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u(D)\alpha} & & \downarrow{\scriptstyle \varphi} \\
B & \xrightarrow{u(D)f'} & I(D) & \xrightarrow{\pi} & B' \\
\| & & \downarrow{\scriptstyle \nu'} & & \downarrow{\scriptstyle \nu} \\
B & \xrightarrow[u(B)]{} & I(B) & \xrightarrow[\pi(B)]{} & TB
\end{array}
\qquad
\begin{array}{ccccc}
B & \xrightarrow{u(B)} & I(B) & \xrightarrow{\pi(B)} & TB \\
\| & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu} \\
B & \xrightarrow{u(D)f'} & I(D) & \xrightarrow{\pi} & B' \\
{\scriptstyle f'}\downarrow & & \| & & \downarrow{\scriptstyle \psi} \\
D & \xrightarrow[u(D)]{} & I(D) & \xrightarrow[\pi(D)]{} & TD
\end{array}$$

constructed in the usual way, to determine representatives $T(f) = \nu\varphi$ and $T(f') = \psi\mu$.

Now we have the data to construct the diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{f'} & D & \xrightarrow{f''} & TA \\
\| & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle j} & & \| \\
A & \xrightarrow{h} & C & \xrightarrow{h'} & F & \xrightarrow{h''} & TA \\
 & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle T(f)} \\
 & & E & = & E & \xrightarrow[\nu g'']{} & TB \\
 & & {\scriptstyle \nu g''}\downarrow & & \downarrow{\scriptstyle j''} & & \\
 & & TB & \xrightarrow[T(f')]{} & TD. & &
\end{array}$$

We just need to check that this diagram commutes. Reading from top left to bottom right: Square 1 commutes as $h = gf$ by definition. Square 2 commutes as j was constructed such that $jf' = h'g$ and $j\alpha = \gamma$. Square 3 commutes since

$$\begin{aligned}
(h''j - f'')\alpha &= h''\gamma - \pi(A) = 0 \\
(h''j - f'')f' &= h''h'g - f''f' = 0 - 0 = 0
\end{aligned}$$

and D is a pushout. Square 4 commutes by construction of j'. Square 5 commutes as

$$\begin{aligned}
(T(f)h'' - \nu g''j')\gamma &= \nu\varphi h''\gamma - \nu g''j'\gamma = \nu(\varphi\pi(A) - g''\beta u(D)\alpha) = \nu(\pi u(D)\alpha - \pi u(D)\alpha) = 0 \\
(T(f)h'' - \nu g''j')h' &= T(f)h''h' - \nu g''g' = 0 - 0 = 0
\end{aligned}$$

and F is a pushout. The last square commutes in $\underline{\mathcal{A}}$. Note,

$$\begin{aligned}(\psi g'' - j'')\beta &= \psi g''\beta - j''\beta = \psi\pi - \pi(D) = 0 \\ (\psi g'' - j'')g' &= \psi g'' g' - j'' g' = 0 - j'' j' h' = 0 - 0 = 0\end{aligned}$$

shows $\psi g'' = j''$ as E is a pushout. Hence, in $\underline{\mathcal{A}}$,

$$T(f')\underline{\nu}\ \underline{g''} - \underline{j''} = \underline{\psi}\ \underline{\mu}\ \underline{\nu}\ \underline{g''} - \underline{j''} = \underline{\psi}\ \underline{g''} - \underline{j''} = 0$$

showing square 6 commutes in $\underline{\mathcal{A}}$. This verifies the octohedral axiom. □

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