SIMPLE ROOTS, CARTAN MATRICES AND DYNKIN DIAGRAMS

SNOW CLOSURE LECTURE NOTES FOR 581C

Abstract. We show that if Φ is an irreducible root system then at most two different root lengths can occur in Φ. We define Cartan matrix and Dynkin diagram associated to a root system and state the Classification theorem to be proven next week. The material covered can be found in [H], sections 10.4, 11.1, 11.2, 11.3, 11.4.

2. Simple Roots, continuation

We fix a base $\Pi \subset \Phi$, $\Pi = \{\alpha_1, \dots, \alpha_n\}$, so that α_i, $1 \le i \le n$, are simple roots.

Definition 2.1 (Partial order on Φ)**.** Let $\alpha, \beta \in \Phi$. We say $\alpha \succ 0$ iff $\alpha \in \Phi_+$, and $\alpha \prec 0$ iff $\alpha \in \Phi_-$. We also say $\alpha \succ \beta$ iff $\alpha - \beta$ is a sum of positive roots (equivalently, simple roots).

Remark 2.2. Note that it can happen that $\alpha \succ \beta$ but $\alpha - \beta$ is NOT a root, only a sum of positive roots. Find an example in B_2 or G_2.

Fact 2.1. *Suppose Φ is irreducible. Then there exists a unique maximal root with respect to the partial order $\succ$. Moreover, if $\alpha = \sum c_i\alpha_i$ then all coefficients c_i are non-zero.*

Definition 2.3. Let $\alpha = \sum c_i\alpha_i$ be a root. Then $\operatorname{ht}\alpha = \sum c_i$ is called the *height* of α.

Exercise*. Determine maximal roots for all irreducible root systems of rank 2.

Remark 2.4. Recall from last time that any root is an image of a simple root under the action of the Weyl group.

Lemma 2.5. *Let Φ be irreducible. Then E does not have non-trivial proper W-invariant subspaces (that is, E is an irreducible representation of W).*

Proof. Proof by Contradiction. Suppose $E_1 \subset E$ is a proper, non-trivial W-invariant subspace. Let $E_1^\perp$ be the orthogonal complement to E_1 with respect to the bilinear form we have on E. Then $E = E_1 \times E_1^\perp$. I claim that for any root α we have that either $\alpha \in E_1$ or $\alpha \in E_1^\perp$.

Suppose $\alpha \notin E_1$. Let P_α be the hyperplane perpendicular to α. We'll show that $E_1 \subset P_\alpha$. Suppose not. Since E_1 is W-invariant, we have $\sigma_\alpha(E_1) = E_1$. If there exists $\lambda \in E_1$ such that $\lambda \notin P_\alpha$, then we have that

$$\sigma(\lambda) - \lambda = -\langle \lambda, \alpha \rangle \alpha \in E_1.$$

Since $\lambda \notin P_\alpha$, we have that $\langle \lambda, \alpha \rangle \neq 0$. Therefore, a non-zero multiple of α is in E_1, and, hence, α itself is in E_1. This contradicts our assumption and we conclude that $E_1 \subset P_\alpha$. This, in turn, implies that $\alpha \perp E_1$. Hence, $\alpha \in E_1^\perp$. This proves the claim.

Let $\Phi_1 = \{\alpha \in E_1\}$ and $\Phi_2 = \{\alpha \in E_1^\perp\}$. We just showed that $\Phi = \Phi_1 \sqcup \Phi_2$ which contradicts irreducibility of Φ. □

Proposition 2.6. *Let Φ be an irreducible root system. Then at most two different root lengths occur in Φ.*

Proof. Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Since W acts irreducibly on E by the lemma, we have that the orbit of α, $W\alpha$, generates E as a vector space. This implies that we can find $w \in W$ such that $(w(\alpha), \beta) \neq 0$. Replacing α with $w(\alpha)$ or $-w(\alpha)$ (which does not change the length), we can assume that $(\alpha, \beta) > 0$.

Date: November 24, 2010.

Recall the table we constructed after making "Observation 1" in this chapter that noted that $\langle \alpha, \beta \rangle = 0, \pm 1, \pm 2, \pm 3$ (we assumed for the table that $||\beta|| \geq ||\alpha||$):

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	ϕ	$\cos \phi$	$\frac{\vert\vert\beta\vert\vert}{\vert\vert\alpha\vert\vert}$
0	0	$\pi/2$	0	??
1	1	$\pi/3$	$1/2$	1
-1	-1	$2\pi/3$	$-1/2$	1
1	2	$\pi/4$	$1/\sqrt{2}$	$\sqrt{2}$
-1	-2	$3\pi/4$	$-1/\sqrt{2}$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}/2$	$\sqrt{3}$
-1	-3	$5\pi/6$	$-\sqrt{3}/2$	$\sqrt{3}$

It follows that $\frac{||\beta||^2}{||\alpha||^2}$ takes values $1, 2, 3, 1/2, 1/3$. Fix α and go over all roots β in Φ.

1. Suppose we find β_1, β_2 such that $\frac{||\beta_1||^2}{||\alpha||^2} = 2$ or $1/2$ and $\frac{||\beta_2||^2}{||\alpha||^2} = 3$ or $1/3$. Then $\frac{||\beta_1||^2}{||\beta_2||^2} = \frac{2}{3}, \frac{3}{2}, 6$ or $\frac{1}{6}$. But from our table we know that this is not possible.

2. Now suppose $\frac{||\beta_1||^2}{||\alpha||^2} = 2$ and $\frac{||\beta_2||^2}{||\alpha||^2} = \frac{1}{2}$. Then $\frac{||\beta_1||^2}{||\beta_2||^2} = 4$. Contradiction again. Similarly for 3 and 1/3.

We conclude that the values of $\frac{||\beta_1||^2}{||\alpha||^2}$ can be 1 and at most one more value from $\{2, 1/2, 3, 1/3\}$. Therefore, only one root length besides $||\alpha||$ can occur. □

Remark 2.7. With just a little more work one can show that all roots of the same length are conjugate under the action of W.

Definition 2.8. In an irreducible root system shorter roots are called *short*, and longer roots are called *long*.

Remark 2.9. One can show that the maximal root is always long (check for yourself for all rank 2 cases)

3. Cartan Matrices and Dynkin Diagrams

As before, we fix a base $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of Φ. Moreover, we now also fix the order of simple roots.

Definition 3.1. Let $a_{ij} = \langle \alpha_i, \alpha_j \rangle$. The matrix (a_{ij}) is called the *Cartan matrix* of Φ.

Remark 3.2. The Cartan matrix is non-singular. We have essentially proved this earlier at some point. Please convince yourself that it follows from non-degeneracy of the bilinear form.

Example 3.3. Here are the Cartan matrices for rank 2 root systems. The ones for B_2 and G_2 depend on the order of simple roots. Figure out which order was chosen for each one (with respect to the labels α, β on the pictures that we had on the board).

$$A_1 \times A_1 \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad A_2 \quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad B_2 \quad \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \qquad G_2 \quad \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Theorem 3.4. *The Cartan matrix determines the root system Φ up to isomorphism.*

We skip the proof of this theorem which can be found in [H, 11.1]. It should be more intuitively clear if we formulate the statement as follows:

Let (Φ, E), (Φ', E') be two root systems of the same rank with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, $\Pi' = \{\alpha'_1, \ldots, \alpha'_n\}$ respectively. Suppose $a_{ij} = a'_{ij}$ for all pairs i, j. Then the map

$$\alpha_i \mapsto \alpha'_i$$

extends uniquely to an isomorphism of root systems $\Phi \simeq \Phi'$.

It is not too hard to reconstruct the root system from its Cartan matrix. Humphreys describes how to do this inductively on page 56.

Definition 3.5 (Coxeter graph.)**.** Let Φ be a root system of rank n, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be simple roots, (a_{ij}) be the Cartan matrix. The *Coxeter graph* of Φ is a graph on n vertices labeled with $\alpha_1, \ldots, \alpha_n$ such that the number of edges connecting α_i and α_j equals $a_{ij}a_{ji} = \langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$.

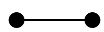
Remark 3.6. There can be **at most** three edges between any two vertices of a Coxeter graph associated to Φ.

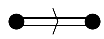
Note that if α_i, α_j are connected by exactly one edge then they must have the same length (since $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle = 1$), and, vice versa, if they have the same length, they are connected by at most on edge. So vertices marked by simple roots of different length are connected by 2 or 3 edges. This, in particular, implies that if the Coxeter graph is simple, it determines completely the Cartan matrix and the root system.

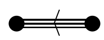
Definition 3.7 (Dynkin diagram)**.** The *Dynkin diagram* of a root system Φ is a partially directed Coxeter graphs with directions assigned to certain arrows as follows: if α_i and α_j are connected by two or three edges, then we put a direction on the edges going from the long to the short root.

Example 3.8. Dynkin diagrams in rank 2.

$\mathbf{A_1 \times A_1}$ 

$\mathbf{A_2}$ 

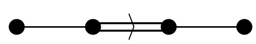
$\mathbf{B_2}$ 

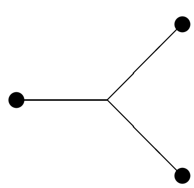
$\mathbf{G_2}$ 

Proposition 3.9. *Dynkin diagram completely determines the Cartan matrix.*

Proof. Exercise □

Example-Exercise. Reconstruct the Cartan matrices for Dynkin diagrams of the types F_4 and D_4:

$\mathbf{F_4}$ 

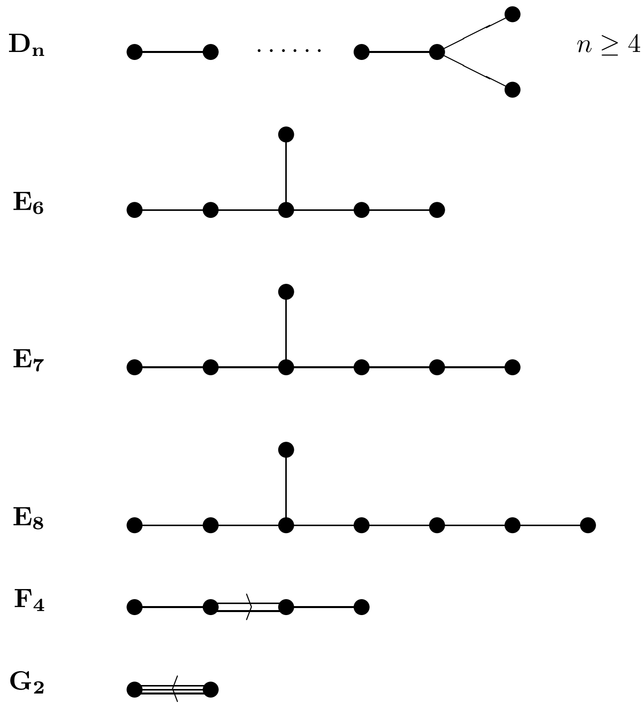
$\mathbf{D_4}$ 

Theorem 3.10 (Cartan-Killing classification)**.** *Let Φ be an irreducible root system of rank n. Then its Dynkin diagram must be one of the following:*

$\mathbf{A_n}$  $n \geq 1$

$\mathbf{B_n}$  $n \geq 2$

$\mathbf{C_n}$  $n \geq 3$



Moreover, root systems corresponding to different diagrams are pair-wise non-isomorphic.

Remark 3.11. In [H2], an abstract Coxeter graph is defined as a *simple* graph with labels assigned to edges. To make the transition, we have to replace double edges with single edges labeled with 4 and triple edges with single edges labeled with 6. Note that this is a very meaningful labeling. In each case, if the label is m, then the angle between the corresponding simple roots equals $\pi - \frac{\pi}{m}$.

References

[H] J. Humphreys, "Introduction to Lie Algebras and Representation Theory".
[H2] J. Humphreys, "Reflection groups and Coxeter groups".