Homework 3 for "Algebraic Structures I", Autumn 2010
due Friday, Nov. 19

For this homework assignment assume that k is an algebraically closed field of characteristic 0.

Problem 1.

(1) Let V be a representation of the Lie algebra $\mathfrak{g}$, and let $W \subset V$ be a subrepresentation. Let $B_v : \mathfrak{g} \times \mathfrak{g} \to k$ be a bilinear form associated to V via the formula $B_V(x, y) = tr(\rho_V(x)\rho_V(y))$, and define B_W and $B_{V/W}$ similarly. Show that $B_V(x, y) = B_W(x, y) + B_{V/W}(x, y)$.

(2) Let $I \subset \mathfrak{g}$ be an ideal in $\mathfrak{g}$. Show that the restriction of the Killing form of $\mathfrak{g}$ to I coincides with the Killing form of I.

Problem 2. Let n be an even number, and let V be a representation of $sp(n)$ obtained by restricting the standard representation of gl_n (that is, V is an n-dimensional vector space, and the action is given by matrix multiplication). Show that B_V as defined in Problem 1 is non-degenerate. (Note that if x, y are two symplectic matrices, then $B_V(x, y)$ is simply $tr(xy)$).

Problem 3. Let $\mathfrak{g}$ be a simple Lie algebra. Show that an invariant symmetric bilinear form on $\mathfrak{g}$ is unique up to a scalar.

Problem 4. Let $\mathfrak{g}$ be a reductive Lie algebra. Show that $\mathfrak{g} \simeq Z(\mathfrak{g}) \oplus \mathfrak{g}'$ where $\mathfrak{g}'$ is semi-simple.