Homework 1 for "Algebraic Structures I", Autumn 2010
due Friday, October 8

Problem 1. Let $A = k[x_1, \dots, x_n]$.

(1) Show that any derivation of A has the form $f_1(\underline{x})\frac{\partial}{\partial x_1} + \ldots + f_n(\underline{x})\frac{\partial}{\partial x_n}$ where $f_1(\underline{x}), \dots, f_n(\underline{x}) \in k[x_1, \dots, x_n]$.

(2) Let $D_f = f_1(\underline{x})\frac{\partial}{\partial x_1} + \ldots + f_n(\underline{x})\frac{\partial}{\partial x_n}$, $D_g = g_1(\underline{x})\frac{\partial}{\partial x_1} + \ldots + g_n(\underline{x})\frac{\partial}{\partial x_n}$. Find the formula for $[D_f, D_g]$.

Problem 2. Let k be an infinite field, and think of $\mathbb{G}_m$ is an algebraic group over k with the coordinate algebra $k[t, \frac{1}{t}]$. Show that $\mathrm{Lie}(\mathbb{G}_m) \simeq g_a$.

Problem 3. Let V be an n-dimensional vector space over k. Let $S^*(V) = \bigoplus_{d=0}^{\infty} S^d(V)$ be the symmetric algebra of V. We have

(i) $S^d(V) \simeq k[x_1, \dots, x_n]_{(d)}$, homogeneous polynomials of degree d

(ii) $S^*(V) \simeq k[x_1, \dots, x_n]$.

Hence, $k[x_1, \dots, x_n]$ has a structure of a representation of $gl(V) \simeq gl_n$ via the standard action of $gl(V)$ on $S^d(V)$. Call this representation $\rho_1 : gl_n \to gl(k[x_1, \dots, x_n])$.

Consider an embedding $gl_n \to \mathrm{Der}_k(k[x_1, \dots, x_n])$ defined by

$$||a_{ij}|| \mapsto \sum a_{ij} x_i \frac{\partial}{\partial x_j}$$

I. Show that this is an embedding of Lie algebras. Conclude that by restricting the action of $\mathrm{Der}_k(k[x_1, \dots, x_n])$ on $k[x_1, \dots, x_n]$ to gl_n via this embedding, we get another representation of gl_n on $k[x_1, \dots, x_n]$. Call it $\rho_2 : gl_n \to gl(k[x_1, \dots, x_n])$.

II. Show that representations ρ_1 and ρ_2 are isomorphic.

Problem 4. Let e, f, h be the standard basis of sl_2, and let $\mathrm{ad} : sl_2 \to gl_3$ be the adjoint representation of sl_2 with respect to the standard basis. Calculate $\mathrm{ad}\, e$, $\mathrm{ad}\, f$ and $\mathrm{ad}\, h$. Is this representation faithful? Is it irreducible?