Problem 1. Let $A = k[x_1, \ldots, x_n]$.

1. Show that any derivation of $A$ has the form $f_1(x) \frac{\partial}{\partial x_1} + \ldots + f_n(x) \frac{\partial}{\partial x_n}$ where $f_1(x), \ldots, f_n(x) \in k[x_1, \ldots, x_n]$.

2. Let $D_f = f_1(x) \frac{\partial}{\partial x_1} + \ldots + f_n(x) \frac{\partial}{\partial x_n}$, $D_g = g_1(x) \frac{\partial}{\partial x_1} + \ldots + g_n(x) \frac{\partial}{\partial x_n}$. Find the formula for $[D_f, D_g]$.

Problem 2. Let $k$ be an infinite field, and think of $\mathbb{G}_m$ is an algebraic group over $k$ with the coordinate algebra $k[t, \frac{1}{t}]$. Show that $\text{Lie}(\mathbb{G}_m) \simeq \mathfrak{g}_a$.

Problem 3. Let $V$ be an $n$-dimensional vector space over $k$. Let $S^* (V) = \bigoplus_{d=0}^{\infty} S^d(V)$ be the symmetric algebra of $V$. We have

(i) $S^d(V) \simeq k[x_1, \ldots, x_n]_{(d)}$, homogeneous polynomials of degree $d$
(ii) $S^*(V) \simeq k[x_1, \ldots, x_n]$.

Hence, $k[x_1, \ldots, x_n]$ has a structure of a representation of $gl(V) \simeq gl_n$ via the standard action of $gl(V)$ on $S^d(V)$. Call this representation $\rho_1 : gl_n \to gl(k[x_1, \ldots, x_n])$.

Consider an embedding $gl_n \to \text{Der}_k(k[x_1, \ldots, x_n])$ defined by $||a_{ij}|| \mapsto \sum a_{ij} x_i \frac{\partial}{\partial x_j}$

I. Show that this is an embedding of Lie algebras. Conclude that by restricting the action of $\text{Der}_k(k[x_1, \ldots, x_n])$ on $k[x_1, \ldots, x_n]$ to $gl_n$ via this embedding, we get another representation of $gl_n$ on $k[x_1, \ldots, x_n]$. Call it $\rho_2 : gl_n \to gl(k[x_1, \ldots, x_n])$.

II. Show that representations $\rho_1$ and $\rho_2$ are isomorphic.

Problem 4. Let $e, f, h$ be the standard basis of $sl_2$, and let $\text{ad} : sl_2 \to gl_3$ be the adjoint representation of $sl_2$ with respect to the standard basis. Calculate $\text{ad} e$, $\text{ad} f$ and $\text{ad} h$. Is this representation faithful? Is it irreducible?