**Homework 2 for "Algebraic Structures I", Autumn 2016**
due Friday, Nov. 28

For this homework assignment assume that $k$ is an algebraically closed field of characteristic 0.

**Problem 1.** Let $A$ be a Hopf algebra, and let $\nabla : A \to A \otimes A$ be the comultiplication. An element $a \in A$ is called primitive if $\nabla(a) = a \otimes 1 + 1 \otimes a$. The subspace of all primitive elements is denoted $Prim(A)$. Show that $Prim(U(g)) = g$ for any Lie algebra $g$ over a field $k$ of characteristic 0.

**Problem 2.** Let $x, y$ be commuting semisimple elements in $gl_n$. Show that $x + y$ is semisimple. Give an example when this fails if $x, y$ do not commute.

**Problem 3.**

(1) Let $V$ be a representation of a Lie algebra $\mathfrak{g}$, and let $W \subset V$ be a subrepresentation. Let $B_V : \mathfrak{g} \times \mathfrak{g} \to k$ be a bilinear form associated to $V$ via the formula $B_V(x, y) = tr(\rho_V(x)\rho_V(y))$, and define $B_W$ and $B_{V/W}$ similarly. Show that $B_V(x, y) = B_W(x, y) + B_{V/W}(x, y)$.

(2) Let $I \subset \mathfrak{g}$ be an ideal in $\mathfrak{g}$. Show that the restriction of the Killing form of $\mathfrak{g}$ to $I$ coincides with the Killing form of $I$.

**Problem 4.** Let $n$ be an even number, and let $V$ be a representation of $sp(n)$ obtained by restricting the standard representation of $gl_n$ (that is, $V$ is an n-dimensional vector space, and the action is given by matrix multiplication). Show that $B_V$ as defined in Problem 1 is non-degenerate. (Note that if $x, y$ are two symplectic matrices, then $B_V(x, y)$ is simply $tr(xy)$).

**Problem 5.** Let $\mathfrak{g}$ be a simple Lie algebra. Show that an invariant symmetric bilinear form on $\mathfrak{g}$ is unique up to a scalar.