Homework 1 for "Algebraic Structures: Lie algebras", Autumn 2016
due Friday, October 21

Problem 1. Let $A = k[x_1, \dots, x_n]$.

(1) Show that any derivation of A has the form $f_1(\underline{x})\frac{\partial}{\partial x_1} + \dots + f_n(\underline{x})\frac{\partial}{\partial x_n}$ where $f_1(\underline{x}), \dots, f_n(\underline{x}) \in k[x_1, \dots, x_n]$.

(2) Let $D_f = f_1(\underline{x})\frac{\partial}{\partial x_1} + \dots + f_n(\underline{x})\frac{\partial}{\partial x_n}, D_g = g_1(\underline{x})\frac{\partial}{\partial x_1} + \dots + g_n(\underline{x})\frac{\partial}{\partial x_n}$. Find the formula for $[D_f, D_g]$.

Problem 2. Show that $\mathrm{Lie}(\mathbb{G}_m) \simeq g_a$.

Problem 3. Let e, f, h be the standard basis of sl_2, and let $\mathrm{ad} : sl_2 \to gl_3$ be the adjoint representation of sl_2 with respect to the standard basis. Calculate $\mathrm{ad}\, e, \mathrm{ad}\, f$ and $\mathrm{ad}\, h$. Is this representation faithful? irreducible? If not, how does it decompose?

Problem 4. Let $A = k[x_1, \dots, x_n]$. Consider an embedding of gl_n into $\mathrm{Der}_k(A)$ as *linear derivations*:

$$(a_{ij}) \mapsto \Sigma a_{ij} x_i \frac{\partial}{\partial x_j}.$$

Since the Lie algebra $\mathrm{Der}_k(A)$ acts on A, this defines a representation of gl_n on A; moreover, this representation preserves degrees. Show that the induced representation on $k[x_1, \dots, x_n]_{(d)}$, homogeneous polynomials of degree d, coincides with the representation of gl_n on $S^d(\bigoplus kx_i)$ under the standard identification $k[x_1, \dots, x_n]_{(d)} \simeq S^d(\bigoplus kx_i)$.

Problem 5. Let k be an algebraically closed field of characteristic $p > 0$. Show that the highest weight modules $V(0), V(1), \dots, V(p-1)$ form a complete list of irreducible representations of sl_2 over k.