WORKSHEET ON ARTINIAN RINGS WITH PROOFS

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All rings are commutative UNLESS specified otherwise (as in Theorem 1.3). This worksheet pursues two main results on Artinian rings:

(1) A commutative Artinian ring is a Noetherian ring of dimension 0 (Theorem 2.6).
(2) Structure theorem for commutative Artinian rings (Theorem 2.9)

1. Hopkins-Levitzki theorem (Courtesy of Reid)

Definition 1.1. An ideal I of a ring R is *nilpotent* if $I^n = \langle x_1 \cdots x_n \,|\, x_i \in I\rangle = (0)$ for some $n \in \mathbb{Z}$.

Lemma 1.2. *The Jacobson radical $J(R)$ of a commutative Artinian ring R is a nilpotent ideal.*

Proof. Suppose that $J(R)$ is not nilpotent; then $J^n(R) \neq 0$ for all n. Since R is Artinian, $J^k(R) = J^{k+n}(R)$ for all $n \in \mathbb{N}$ for some $k \in \mathbb{N}$. Now, let $\Sigma = \{\mathfrak{a} \subseteq R \,|\, J^k(R)\mathfrak{a} \neq 0\}$. By assumption Σ is nonzero since $J(R)$ is not nilpotent. If we order Σ my reverse containment, the Artinian condition guarantees that any chain has an upper bound and hence a maximal element by Zorn's lemma (which is in fact minimal with respect to inclusion); call it $\mathfrak{h}$. Now, if $h \in \mathfrak{h}$ is not annihilated by $J^k(R)$ (which, since $\mathfrak{h} \in \Sigma$ must exist) then $\langle h\rangle \subseteq \mathfrak{h}$ and by inclusion-minimality $\mathfrak{h} = \langle h\rangle$. Now, $J^k(R)(\mathfrak{h}) = J^k(R)\langle h\rangle \subseteq \langle h\rangle$ and by the stability $J^{k+n}(R) = J^kR$ we have that this ideal is itself nonzero when multiplied by $J^k(R)$ and lies in Σ; by inclusion minimality $\langle h\rangle = J^k(R)\langle h\rangle$. But then $\langle h\rangle = J(R)\langle h\rangle$ and is finitely generated as an R-module, so we can apply Nakayama's lemma to get that $\langle h\rangle = 0$, contradicting its inclusion in Σ. Hence $J(R)$ is nilpotent. □

The result of the lemma holds for non-commutative rings. The proof should work almost without changeonce one makes the necessary "non-commutativity" adjustements. For the next theorem, we will assume that the result of the lemma holds for not necessarily commutative rings.

Theorem 1.3. *[Hopkins-Levitzki theorem] Let R be an Artinian ring (not necessarily commutative), and M be a finitely generated R-module. Prove that M is a Noetherian R-module.*

Proof. Because the Jacobson radical is nilpotent, we have a chain of R-modules

$$0 = J^n(R)M \subseteq J^{n-1}(R)M \subseteq \cdots \subseteq J(R)M \subseteq M$$

We do induction along this series to show that M is Noetherian. Clearly 0 is Noetherian, so our base case is established. Now, suppose that $J^k(R)M$ is Noetherian; we wish to show that $J^{k-1}(R)M$ is Noetherian. Consider the exact sequence

$$0 \to J^k(R)M \to J^{k-1}(R)M \to J^{k-1}(R)M/J^k(R)M \to 0$$

with the obvious inclusion and projection maps. We know that $J^k(R)M$ is Noetherian by inductive hypothesis, so if we can show that $J^{k-1}(R)M/J^k(R)M$ is Noetherian then since we have an exact sequence we can conclude that $J^{k-1}(R)M$ is Noetherian. Now, $J^{k-1}(R)M/J^k(R)M$ inherits an $R/J(R)$-module structure since $J(R)$ annihilates $J^{k-1}(R)M/J^k(R)M$. As $R/J(R)$ is a semisimple ring (which uses the Artinian condition of R; $R/J(R)$ is Artinian since R is), $J^{k-1}(R)M/J^k(R)M$ is a direct sum of simple $R/J(R)$ modules. Since M is finitely generated M is Artinian. Being a submodule of M, $J^{k-1}(R)M$ is also Artinian and so is the quotient module $J^{k-1}(R)M/J^k(R)M$. In order to satisfy the descending chain condition, $J^{k-1}(R)M/J^k(R)M$ must have only finitely many summands in its decomposition. But this means that $J^{k-1}(R)M/J^k(R)M$ has a composition series and is therefore Noetherian. Then $J^{k-1}(R)M$ is Noetherian and our inductive step is done. As desired, by induction, M is a Noetherian R-module. □

Corollary 1.4. *An Artinian ring is Noetherian.*

2. Krull dimension of commutative Artinian rings (courtesy of Jim)

Lemma 2.1. *Let R be an Artinian integral domain. Then R is a field.*

Proof. Let $x \in R$ be nonzero and consider $(x) \supseteq (x^2) \supseteq \dots$ which must stabilize because R is Artinian. Then $(x^{n+1}) = (x^n)$ for some $n > 0$ so write $x^n = ax^{n+1}$ for some $a \in R$. As R is an integral domain and x is nonzero we can cancel x^n and conclude that $1 = ax$, so x is a unit. □

Proposition 2.2. *Let R be an Artinian ring. Then any prime ideal is maximal.*

Proof. Let $\mathfrak{p}$ be prime, then $R/\mathfrak{p}$ is an Artinian integral domain, hence it is a field which proves that $\mathfrak{p}$ is maximal. □

Corollary 2.3. *Let R be an Artinian ring. Then the Krull dimension of R is zero.*

Proof. Every prime ideal is maximal so there cannot be a chain of prime ideals of positive length. □

Proposition 2.4. *Let R be an Noetherian ring. Then $\mathfrak{N}(R)$ is a nilpotent ideal.*

Proof. As R is Noetherian let $x_1, \dots, x_n$ be generators of $\mathfrak{N}(R)$. Each of these elements is nilpotent so we may choose $k \in \mathbb{N}$ large enough so that $x_i^k = 0$ for each i. An element of $\mathfrak{N}(R)$ can be written as $y = a_1x_1 + \dots + a_nx_n$. If we multiply nk such elements together the result will be a linear combination of monomials of the form $ax_1^{i_1} \cdots x_n^{i_n}$ where $i_1 + \dots + i_n = nk$. By the generalized pigeonhole principle we must have $i_j \geq k$ for some j, hence the result of the multiplication is zero. This gives $\mathfrak{N}(R)^{nk} = 0$. □

Lemma 2.5. *(1). Let $\mathfrak{p}$ be a prime ideal in R. Then* $\operatorname{rad}(\mathfrak{p}^n) = \mathfrak{p}$.
(2). Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals in R which are also relatively prime. Then $\mathfrak{p}_1^n, \mathfrak{p}_2^m$ are relatively prime for any $n, m > 0$.

Proof. The first is a previous homework problem. For the second note that for any ideal I and J we have

$$V(I + J) = V(\langle I \cup J \rangle) = V(I \cup J) = V(I) \cap V(J)$$

and for a prime ideal $\mathfrak{p}$ we have $V(\mathfrak{p}^k) = V(\operatorname{rad}\mathfrak{p}^k) = V(\mathfrak{p})$ therefore

$$V(\mathfrak{p}_1^n + \mathfrak{p}_2^m) = V(\mathfrak{p}_1^n) \cap V(\mathfrak{p}_2^m) = V(\mathfrak{p}_1) \cap V(\mathfrak{p}_2) = V(\mathfrak{p}_1 + \mathfrak{p}_2) = V(R) = \emptyset.$$

If $\mathfrak{p}_1^n + \mathfrak{p}_2^m$ were a proper ideal in R it would be contained in a maximal ideal. Maximal ideals are prime therefore $V(\mathfrak{p}_1^n + \mathfrak{p}_2^m)$ would be nonempty, which it is not, hence $\mathfrak{p}_1^n + \mathfrak{p}_2^m = R$. □

Theorem 2.6. *A ring R is Artinian if and only if it is Noetherian of Krull dimension* 0.

Proof. Artinian implies Noetherian by the Hopkins-Levitzki theorem and the dimension is 0 by Corollary 2.3. Now let R be a zero-dimensional Noetherian ring. Every maximal ideal is prime; since the dimension is zero every prime ideal is both a maximal ideal and a minimal prime ideal. By Problem 2 from Homework 2, R has finitely many minimal prime ideals, hence finitely many maximal ideals.

Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n\}$ be the set of all maximal ideals in R. This is also the set of all prime ideals in

R therefore $\mathfrak{N} = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ (because distinct maximal ideals are relatively prime). Hence, by Proposition 2.4, there exists a $k > 0$ such that $\mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = 0$. By Lemma 1.5 $\mathfrak{m}_1^k, \ldots, \mathfrak{m}_n^k$ are pairwise relatively prime so the Chinese Remainder Theorem gives

$$R = R/\mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = R/\mathfrak{m}_1^k \times \cdots \times R/\mathfrak{m}_n^k.$$

Each $R/\mathfrak{m}_i^k$ is a local ring. To see this note that the maximal ideals of $R/\mathfrak{m}_i^k$ correspond to maximal ideals of R that contain $\mathfrak{m}_i^k$. As maximal ideals are prime this means maximal ideals that contain $\mathfrak{m}_i$, of which there is only one, $\mathfrak{m}_i$.

The above shows not only that $R/\mathfrak{m}_i^k$ is local, but that the maximal ideal of this ring is nilpotent. By Corollary 1.7 we conclude that $R/\mathfrak{m}_i^k$ is Artinian and by Lemma 1.8 we find that R itself is Artinian (we will not use the conclusions of this last paragraph in those proofs). □

Corollary 2.7. *Let R be a Noetherian local ring with a maximal ideal $\mathfrak{m}$. Then one of the following holds:*
(•) *either $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for any $n > 0$*
(•) *or there exists n such that $\mathfrak{m}^n = 0$. In the latter case, R is Artinian.*

Proof. Assume the first is not the case and there exists an $n > 0$ such that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. As R is Noetherian $\mathfrak{m}^n$ is finitely generated and as R is local $J(R) = \mathfrak{m}$. Then the Nakayama lemma gives $\mathfrak{m}^n = 0$. When this is the case consider the chain

$$0 = \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1} \subseteq \ldots \subseteq \mathfrak{m}^2 \subseteq \mathfrak{m} \subseteq R.$$

Trivially $\mathfrak{m}^n$ is Artinian. If $\mathfrak{m}^{k+1}$ is Artinian then observe that $\mathfrak{m}^k$ is an R-module therefore $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is an $R/\mathfrak{m}$-module. As $\mathfrak{m}$ is maximal $R/\mathfrak{m}$ is a field so $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a vector space. Because R is Noetherian $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is as well; therefore, it is a vector space of finite dimension, hence it is Artinian. Because $\mathfrak{m}^k$ is an extension of $\mathfrak{m}^{k+1}$ by $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ and both of these are Artinian we conclude that $\mathfrak{m}^k$ is Artinian. By induction R is Artinian. □

In other words, a local Noetherian ring is Artinian if and only if the unique maximal ideal is nilpotent.

Lemma 2.8. *Let R_1, R_2 be Artinian rings. Then $R_1 \times R_2$ is also Artinian.*

Proof. We have previously shown that R_1 and R_2 can be made $R_1 \times R_2$-modules via the actions $(a, b) \cdot c = ac$ and $(a, b) \cdot c = bc$ respectively. It is clear from the definition of the action that R_i has the same submodules as an $R_1 \times R_2$-module that it does as an R_i-module so R_1 and R_2 are Artinian $R_1 \times R_2$-modules. From the obvious short exact sequence $R_1 \to R_1 \times R_2 \to R_2$ we find that $R_1 \times R_2$ is an Artinian $R_1 \times R_2$-module and hence an Artinian ring. □

Theorem 2.9. *Any Artinian ring decomposes uniquely (up to isomorphism) as a direct product of finitely many local Artinian rings.*

Proof. Existence of this decomposition is given in the proof of Theorem 1.6 so we need only show that this decomposition is unique. Assume $R = A_1 \times \cdots \times A_l$ and each A_i is a local ring with maximal ideal $\mathfrak{a}_i$. Every ideal in R is of the form $I_1 \times \cdots \times I_l$ where I_j is an ideal in A_j. For an arbitrary maximal ideal $\mathfrak{m}_i = I_1 \times \cdots \times I_l$ one of the I_j must be proper and hence contained in $\mathfrak{a}_j$.

Then $\mathfrak{m}_i$ is contained in and therefore equal to $A_1 \times \cdots \times \mathfrak{a}_j \times \cdots \times A_l$. This proves that $l = n$ and

$$\begin{gathered} \mathfrak{a}_1 \times A_2 \times \cdots \times A_n \\ A_1 \times \mathfrak{a}_2 \times \cdots \times A_n \\ \vdots \\ A_1 \times \cdots \times A_{l-1} \times \mathfrak{a}_n \end{gathered}$$

is a complete list of maximal ideals. Without loss of generality we assume that the $\mathfrak{a}_i$ are ordered so that $\mathfrak{m}_i = A_1 \times \cdots \times \mathfrak{a}_i \times \cdots \times A_n$. From this we have $\mathfrak{m}_i^k = A_1 \times \cdots \times \mathfrak{a}_i^k \times \cdots \times A_n$ and $0 = \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = \mathfrak{a}_1^k \times \cdots \times \mathfrak{a}_n^k$ therefore $\mathfrak{a}_i^k = 0$ for each i. Finally

$$\begin{aligned} R/\mathfrak{m}_i^k &= (A_1 \times \cdots \times A_n)/(A_1 \times \cdots \times 0 \times \cdots \times A_n) \\ &= A_1/A_1 \times \cdots \times A_i/0 \times \cdots \times A_n/A_n \\ &= 0 \times \cdots \times A_i \times \cdots \times 0 \\ &= A_i. \end{aligned}$$

This proves that the decomposition is unique. □

Remark 2.10. For an Artinian ring R, $\operatorname{Spec} R$ is just a union of finitely many points. Zariski topology becomes a discrete topology. $\operatorname{Spec} R$ is irreducible if and only if R is local.