All rings are commutative UNLESS specified otherwise (as in Theorem 1.3). This worksheet pursues two main results on Artinian rings:

(1) A commutative Artinian ring is a Noetherian ring of dimension 0 (Theorem 2.6).
(2) Structure theorem for commutative Artinian rings (Theorem 2.9)

1. Hopkins-Levitzki theorem (Courtesy of Reid)

**Definition 1.1.** An ideal $I$ of a ring $R$ is nilpotent if $I^n = \langle x_1 \cdots x_n \mid x_i \in I \rangle = (0)$ for some $n \in \mathbb{Z}$.

**Lemma 1.2.** The Jacobson radical $J(R)$ of a commutative Artinian ring $R$ is a nilpotent ideal.

**Proof.** Suppose that $J(R)$ is not nilpotent; then $J^n(R) \neq 0$ for all $n$. Since $R$ is Artinian, $J^k(R) = J^{k+n}(R)$ for all $n \in \mathbb{N}$ for some $k \in \mathbb{N}$. Now, let $\Sigma = \{a \subseteq R \mid J^k(R)a \neq 0\}$. By assumption $\Sigma$ is nonzero since $J(R)$ is not nilpotent. If we order $\Sigma$ my reverse containment, the Artinian condition guarantees that any chain has an upper bound and hence a maximal element by Zorn’s lemma (which is in fact minimal with respect to inclusion); call it $h$. Now, if $h \in \mathfrak{h}$ is not annihilated by $J^k(R)$ (which, since $\mathfrak{h} \in \Sigma$ must exist) then $\langle h \rangle \subseteq \mathfrak{h}$ and by inclusion-minimality $\mathfrak{h} = \langle h \rangle$. Now, $J^k(R)\langle h \rangle = J^k(R)\langle h \rangle \subseteq \langle h \rangle$ and by the stability $J^{k+n}(R) = J^kR$ we have that this ideal is itself nonzero when multiplied by $J^k(R)$ and lies in $\Sigma$; by inclusion minimality $\langle h \rangle = J^k(R)\langle h \rangle$. But then $\langle h \rangle = J(R)\langle h \rangle$ and is finitely generated as an $R$-module, so we can apply Nakayama’s lemma to get that $\langle h \rangle = 0$, contradicting its inclusion in $\Sigma$. Hence $J(R)$ is nilpotent. \(\hfill \square\)

The result of the lemma holds for non-commutative rings. The proof should work almost without change once one makes the necessary “non-commutativity” adjustments. For the next theorem, we will assume that the result of the lemma holds for not necessarily commutative rings.

**Theorem 1.3.** [Hopkins-Levitzki theorem] Let $R$ be an Artinian ring (not necessarily commutative), and $M$ be a finitely generated $R$-module. Prove that $M$ is a Noetherian $R$-module.

**Proof.** Because the Jacobson radical is nilpotent, we have a chain of $R$-modules

$$0 = J^n(R)M \subseteq J^{n-1}(R)M \subseteq \cdots \subseteq J(R)M \subseteq M$$

We do induction along this series to show that $M$ is Noetherian. Clearly $0$ is Noetherian, so our base case is established. Now, suppose that $J^k(R)M$ is Noetherian; we wish to show that $J^{k-1}(R)M$ is Noetherian. Consider the exact sequence

$$0 \to J^k(R)M \to J^{k-1}(R)M \to J^{k-1}(R)M/J^k(R)M \to 0$$

with the obvious inclusion and projection maps. We know that $J^k(R)M$ is Noetherian by inductive hypothesis, so if we can show that $J^{k-1}(R)M/J^k(R)M$ is Noetherian then since we have an exact sequence we can conclude that $J^{k-1}(R)M$ is Noetherian. Now, $J^{k-1}(R)M/J^k(R)M$ inherits an $R/J(R)$-module structure since $J(R)$ annihilates $J^{k-1}(R)M/J^k(R)M$. As $R/J(R)$ is a semisimple ring (which uses the Artinian condition of $R$; $R/J(R)$ is Artinian since $R$ is), $J^{k-1}(R)M/J^k(R)M$ is a direct sum of simple $R/J(R)$ modules. Since $M$ is finitely generated $M$ is Artinian. Being a submodule of $M$, $J^{k-1}(R)M$ is also Artinian and so is the quotient module $J^{k-1}(R)M/J^k(R)M$. In
order to satisfy the descending chain condition, \( J^{k-1}(R)M/J^k(R)M \) must have only finitely many summands in its decomposition. But this means that \( J^{k-1}(R)M/J^k(R)M \) has a composition series and is therefore Noetherian. Then \( J^{k-1}(R)M \) is Noetherian and our inductive step is done. As desired, by induction, \( M \) is a Noetherian \( R \)-module.

\[ \square \]

**Corollary 1.4.** An Artinian ring is Noetherian.

### 2. Krull dimension of commutative Artinian rings (courtesy of Jim)

**Lemma 2.1.** Let \( R \) be an Artinian integral domain. Then \( R \) is a field.

**Proof.** Let \( x \in R \) be nonzero and consider \( (x) \supseteq (x^2) \supseteq \ldots \) which must stabilize because \( R \) is Artinian. Then \( (x^{n+1}) = (x^n) \) for some \( n > 0 \) so write \( x^n = ax^{n+1} \) for some \( a \in R \). As \( R \) is an integral domain and \( x \) is nonzero we can cancel \( x^n \) and conclude that \( 1 = ax \), so \( x \) is a unit.

**Proposition 2.2.** Let \( R \) be an Artinian ring. Then any prime ideal is maximal.

**Proof.** Let \( p \) be prime, then \( R/p \) is an Artinian integral domain, hence it is a field which proves that \( p \) is maximal.

**Corollary 2.3.** Let \( R \) be an Artinian ring. Then the Krull dimension of \( R \) is zero.

**Proof.** Every prime ideal is maximal so there cannot be a chain of prime ideals of positive length.

**Proposition 2.4.** Let \( R \) be an Noetherian ring. Then \( \mathfrak{M}(R) \) is a nilpotent ideal.

**Proof.** As \( R \) is Noetherian let \( x_1, \ldots, x_n \) be generators of \( \mathfrak{M}(R) \). Each of these elements is nilpotent so we may choose \( k \in \mathbb{N} \) large enough so that \( x_i^k = 0 \) for each \( i \). An element of \( \mathfrak{M}(R) \) can be written as \( y = a_1x_1 + \cdots + a_nx_n \). If we multiply \( nk \) such elements together the result will be a linear combination of monomials of the form \( ax_1^{i_1} \cdots x_n^{i_n} \) where \( i_1 + \cdots + i_n = nk \). By the generalized pigeonhole principle we must have \( i_j \geq k \) for some \( j \), hence the result of the multiplication is zero.

This gives \( \mathfrak{M}(R)^{nk} = 0 \).

**Lemma 2.5.** (1) Let \( p \) be a prime ideal in \( R \). Then \( \text{rad}(p^n) = p \).

(2) Let \( p_1, p_2 \) be prime ideals in \( R \) which are also relatively prime. Then \( p_1^n, p_2^m \) are relatively prime for any \( n, m > 0 \).

**Proof.** The first is a previous homework problem. For the second note that for any ideal \( I \) and \( J \) we have

\[
V(I + J) = V((I \cup J)) = V(I \cup J) = V(I) \cap V(J)
\]

and for a prime ideal \( p \) we have \( V(p^k) = V(\text{rad} p^k) = V(p) \) therefore

\[
V(p_1^n + p_2^m) = V(p_1^n) \cap V(p_2^m) = V(p_1+ p_2) = V(p_1 + p_2) = V(R) = 0.
\]

If \( p_1^n + p_2^m \) were a proper ideal in \( R \) it would be contained in a maximal ideal. Maximal ideals are prime therefore \( V(p_1^n + p_2^m) \) would be nonempty, which it is not, hence \( p_1^n + p_2^m = R \).

**Theorem 2.6.** A ring \( R \) is Artinian if and only if it is Noetherian of Krull dimension 0.

**Proof.** Artinian implies Noetherian by the Hopkins-Levitzki theorem and the dimension is 0 by Corollary 2.3. Now let \( R \) be a zero-dimensional Noetherian ring. Every maximal ideal is prime; since the dimension is zero every prime ideal is both a maximal ideal and a minimal prime ideal. By Problem 2 from Homework 2, \( R \) has finitely many minimal prime ideals, hence finitely many maximal ideals.

Let \( \{m_1, m_2, \ldots, m_n\} \) be the set of all maximal ideals in \( R \). This is also the set of all prime ideals in
$R$ therefore $\mathfrak{m} = m_1 \cap \ldots \cap m_n = m_1 \cdots m_n$ (because distinct maximal ideals are relatively prime). Hence, by Proposition 2.4, there exists a $k > 0$ such that $m_1^k \cdots m_n^k = 0$. By Lemma 1.5 $m_1^k, \ldots, m_n^k$ are pairwise relatively prime so the Chinese Remainder Theorem gives

$$R = \frac{R}{m_1^k} \cdots \frac{R}{m_n^k} = \frac{R}{m_1^k} \times \cdots \times \frac{R}{m_n^k}.$$

Each $\frac{R}{m_i^k}$ is a local ring. To see this note that the maximal ideals of $\frac{R}{m_i^k}$ correspond to maximal ideals of $R$ that contain $m_i^k$. As maximal ideals are prime this means maximal ideals that contain $m_i$, of which there is only one, $m_i$.

The above shows not only that $\frac{R}{m_i^k}$ is local, but that the maximal ideal of this ring is nilpotent. By Corollary 1.7 we conclude that $\frac{R}{m_i^k}$ is Artinian and by Lemma 1.8 we find that $R$ itself is Artinian (we will not use the conclusions of this last paragraph in those proofs).

**Corollary 2.7.** Let $R$ be a Noetherian local ring with a maximal ideal $m$. Then one of the following holds:

- (i) either $m^n \neq m^{n+1}$ for any $n > 0$
- (ii) or there exists $n$ such that $m^n = 0$. In the latter case, $R$ is Artinian.

**Proof.** Assume the first is not the case and there exists an $n > 0$ such that $m^n = m^{n+1}$. As $R$ is Noetherian $m^n$ is finitely generated and as $R$ is local $J(R) = m$. Then the Nakayama lemma gives $m^n = 0$. When this is the case consider the chain

$$0 = m^n \subseteq m^{n-1} \subseteq \cdots \subseteq m^2 \subseteq m \subseteq R.$$

Trivially $m^n$ is Artinian. If $m^{k+1}$ is Artinian then observe that $m^k$ is an $R$-module therefore $m^k/m^{k+1}$ is an $R/m$-module. As $m$ is maximal $R/m$ is a field so $m^k/m^{k+1}$ is a vector space. Because $R$ is Noetherian $m^k/m^{k+1}$ is as well; therefore, it is a vector space of finite dimension, hence it is Artinian. Because $m^k$ is an extension of $m^{k+1}$ by $m^k/m^{k+1}$ and both of these are Artinian we conclude that $m^k$ is Artinian. By induction $R$ is Artinian. □

In other words, a local Noetherian ring is Artinian if and only if the unique maximal ideal is nilpotent.

**Lemma 2.8.** Let $R_1, R_2$ be Artinian rings. Then $R_1 \times R_2$ is also Artinian.

**Proof.** We have previously shown that $R_1$ and $R_2$ can be made $R_1 \times R_2$-modules via the actions $(a, b) \cdot c = ac$ and $(a, b) \cdot c = bc$ respectively. It is clear from the definition of the action that $R_i$ has the same submodules as an $R_1 \times R_2$-module that it does as an $R_i$-module so $R_1$ and $R_2$ are Artinian $R_1 \times R_2$-modules. From the obvious short exact sequence $R_1 \rightarrow R_1 \times R_2 \rightarrow R_2$ we find that $R_1 \times R_2$ is an Artinian $R_1 \times R_2$-module and hence an Artinian ring. □

**Theorem 2.9.** Any Artinian ring decomposes uniquely (up to isomorphism) as a direct product of finitely many local Artinian rings.

**Proof.** Existence of this decomposition is given in the proof of Theorem 1.6 so we need only show that this decomposition is unique. Assume $R = A_1 \times \cdots \times A_l$ and each $A_i$ is a local ring with maximal ideal $a_i$. Every ideal in $R$ is of the form $I_1 \times \cdots \times I_l$ where $I_j$ is an ideal in $A_j$. For an arbitrary maximal ideal $m_i = I_1 \times \cdots \times I_l$ one of the $I_j$ must be proper and hence contained in $a_j$. □
Then \( m_i \) is contained in and therefore equal to \( A_1 \times \cdots \times a_j \times \cdots \times A_l \). This proves that \( l = n \) and
\[
\begin{align*}
& a_1 \times A_2 \times \cdots \times A_n \\
& A_1 \times a_2 \times \cdots \times A_n \\
& \quad \vdots \\
& A_1 \times \cdots \times A_{i-1} \times a_i \times A_{i+1} \times \cdots \times A_n
\end{align*}
\]
is a complete list of maximal ideals. Without loss of generality we assume that the \( a_i \) are ordered so that \( m_i = A_1 \times \cdots \times a_i \times \cdots \times A_n \). From this we have \( m_i^k = A_1 \times \cdots \times a_i^k \times \cdots \times A_n \) and \( 0 = m_i^k \cdots m_n^k = a_1^k \times \cdots \times a_n^k \) therefore \( a_i^k = 0 \) for each \( i \). Finally
\[
\begin{align*}
R/m_i^k &= (A_1 \times \cdots \times A_n)/(A_1 \times \cdots \times 0 \times \cdots \times A_n) \\
&= A_1/A_1 \times \cdots \times A_i/0 \times \cdots \times A_n/A_n \\
&= 0 \times \cdots \times A_i \times \cdots \times 0 \\
&= A_i.
\end{align*}
\]
This proves that the decomposition is unique. \( \square \)

**Remark 2.10.** For an Artinian ring \( R \), Spec \( R \) is just a union of finitely many points. Zariski topology becomes a discrete topology. Spec \( R \) is irreducible if and only if \( R \) is local.