All rings are commutative with 1. This worksheet pursues two main results on Artinian rings:

1. An Artinian ring is a Noetherian ring of dimension 0 (Thm 1.6).
2. Structure theorem for Artinian rings (Thm 1.9)

**Lemma 1.1.** Let $R$ be an Artinian integral domain. Then $R$ is a field.  

**Proof.** Exercise  \( \square \)

**Proposition 1.2.** Let $R$ be an Artinian ring. Then any prime ideal is maximal.

**Proof.** Exercise  \( \square \)

**Corollary 1.3.** Let $R$ be an Artinian ring. Then the Krull dimension of $R$ is zero.

**Proof.** Exercise (one line proof though).  \( \square \)

**Proposition 1.4.** Let $R$ be an Noetherian ring. Then $\mathfrak{N}(R)$ is a nilpotent ideal.

**Proof.** Exercise  \( \square \)

**Lemma 1.5.** (1). Let $p$ be a prime ideal in $R$. Then $\text{rad}(p^n) = p$.
(2). Let $p_1, p_2$ be prime ideals in $R$ which are also relatively prime. Then $p_1^n, p_2^m$ are relatively prime for any $n, m > 0$.

**Proof.** Exercise (use properties of radicals from Hw. 1 for a very short proof of 2)).  \( \square \)

**Theorem 1.6.** A ring $R$ is Artinian if and only if it is Noetherian of Krull dimension 0.

**Proof.** Artinian implies Noetherian by Hopkins-Levitzki theorem (Homework 1, problem 5); dimension is 0 by Cor. 1.3.

Now let $R$ be a zero-dimensional Noetherian ring. By pr. 2, Hm. 2, $R$ has finitely many minimal prime ideals; since dimension is zero, all prime ideals are maximal. Let $\{m_1, m_2, \ldots, m_n\}$ be the set of all maximal ideals in $R$. Then $\mathfrak{N} = m_1 \cap \ldots \cap m_n = m_1 \cdot \ldots \cdot m_n$. Hence, $m_1^\ell \cdot \ldots \cdot m_n^\ell = 0$ for a big enough $\ell$ by Prop. 1.4. Now show that $R$ has a composition series and conclude that it is Artinian. Finish the proof  \( \square \)

**Corollary 1.7.** Let $R$ be a Noetherian local ring with a maximal ideal $m$. Then one of the following holds:

- either $m^n \neq m^{n+1}$ for any $n > 0$
- or there exists $n$ such that $m^n = 0$. In the latter case, $R$ is Artinian.

**Proof.** Exercise  \( \square \)

In other words, a local Noetherian ring is Artinian if and only if the unique maximal ideal is nilpotent.

**Lemma 1.8.** Let $R_1, R_2$ be Artinian rings. Then $R_1 \times R_2$ is also Artinian.
Proof. Exercise

**Theorem 1.9.** Any Artinian ring decomposes uniquely (up to isomorphism) as a direct product of finitely many local Artinian rings.

*Proof.* Exercise (don’t forget Chinese Remainder theorem).

**Remark 1.10.** For an Artinian ring $R$, Spec $R$ is just a union of finitely many points. Zariski topology becomes a discrete topology. Spec $R$ is irreducible if and only if $R$ is local.