**Homework 6 for 506, Spring 2010: (mostly) ABSTRACT NONSENSE**

A is a commutative ring with identity.

**Problem 1.** Let $V, W$ be algebraic sets. Prove that $f : V \to W$ induces an isomorphism between $V$ and a closed subset of $W$ if and only if $f^* : k[W] \to k[V]$ is surjective.

In this case, $\phi$ is called a **closed embedding**.

**Problem 2.** Let $a \subset A$ be an ideal, and $M$ be an $A$-module. Show that $A/a \otimes_A M \cong M/aM$.

**Problem 3.** Prove that the Hom-functor is left exact, and even more:

1. Show that $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $A$-modules if and only if for any $A$-module $N$,

\[ 0 \to \text{Hom}(M'', N) \to \text{Hom}(M, N) \to \text{Hom}(M', N) \]

is exact;

2. Show that $0 \to N' \to N \to N'' \to 0$ is an exact sequence of $A$-modules if and only if for any $A$-module $M$

\[ 0 \to \text{Hom}(M, N') \to \text{Hom}(M, N) \to \text{Hom}(M, N'') \]

is exact.

For both cases give examples showing that Hom is not exact.

**Problem 4.** Calculate:

1. $\text{Hom}_\mathbb{Z}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$

2. $\mathbb{Z}/n\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/m\mathbb{Z}$

**Problem 5.** Recall that for $\phi : M \to N$, $\text{Coker} \phi = N/\text{im} \phi$.

Prove the **Snake Lemma**: for any commutative diagram of $A$-modules with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \rightarrow & 0 \\
& \downarrow{\phi'} & & \downarrow{\phi} & & \downarrow{\phi''} & & \\
0 & \rightarrow & N' & \xrightarrow{t} & N & \xrightarrow{s} & N'' & \rightarrow & 0 \\
\end{array}
\]

the following sequence is exact:

\[ 0 \to \text{Ker} \phi' \to \text{Ker} \phi \to \text{Ker} \phi'' \xrightarrow{\delta} \text{Coker} \phi' \to \text{Coker} \phi \to \text{Coker} \phi'' \to 0. \]

Here,

$\delta : \text{Ker} \phi'' \to \text{Coker} \phi'$

is the **connecting homomorphism** defined as follows. Let $m'' \in \text{Ker} \phi''$. Then there exists $m \in M$ such that $m'' = g(m)$. Commutativity of the diagram together with the exactness of the bottom row imply that $\phi(m) = t(n')$ for some $n' \in N'$. We define $\delta(m'') = \bar{n} \in N'/\text{Im} \phi' = \text{Coker} \phi'$. All the other maps in the sequence are naturally induced by the maps in the diagram.

**Remark 1.** Checking that $\delta$ is well-defined is part of the exercise!

**Remark 2.** This kind of argument is called “diagram chase” - tedious but very useful.