Midterm for 506, Spring 2009

Monday, May 4

You may use you notes and anything we proved in class: just state clearly the fact/theorem you are using.

Each problem is worth 10 points. The maximal score is **40 points**: you only need to do 4 problems (correctly!) to get full marks. If you do all 5 problems, I shall simply drop the lowest score.

Problem 1. An idempotent is an element e such that $e^2 = e$. Every ring has two trivial idempotents: 0 and 1. Prove that a local ring does not have non-trivial idempotents.

Problem 2.

(1) Let k be an algebraically closed field. Describe $\operatorname{Spec} k[t, \frac{1}{t}]$. (Pictures are appreciated.)

(2) Let k be an algebraically closed field of characteristic p. Describe $\operatorname{Spec} \frac{k[t]}{t^p-1}$.

Problem 3. Let $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ be a map given by the formula $\phi(x) = (x, x^2, x^3)$. Prove that $C = \operatorname{Im} \phi$ is an algebraic set, and that $\phi : \mathbb{A}^1 \to C$ is an isomorphism.

Problem 4. Let X, Y be algebraic sets. Prove that $\phi : X \to Y$ induces an isomorphism between X and a closed subset of Y if and only if $\phi^* : k[Y] \to k[X]$ is surjective.

Problem 5. Let A be a principal ideal domain, and M, N be A-modules. Find necessary and sufficient conditions for $M \otimes_A N$ to be zero.