Homework 8 for 506, Spring 2009
due Friday, June 5

Problem 1.[10pt] Let A be a Noetherian local ring, $\mathfrak{m}$ be its maximal ideal and $k = A/\mathfrak{m}$ be the residue field. Let M be a finitely generated A-module. Prove that the following are equivalent:

(1) M is free
(2) M is flat
(3) M is projective
(4) $\mathfrak{m} \otimes M \to A \otimes M$ is injective
(5) $\mathrm{Tor}_1^A(k, M) = 0$

Problem 2.[1pt] Let A be a Noetherian ring, M be a finitely generated A-module. Prove that the following are equivalent:

(1) M is a flat A-module
(2) $M_{\mathfrak{p}}$ is free as an $A_{\mathfrak{p}}$ -module for any prime ideal $\mathfrak{p} \subset A$
(3) $M_{\mathfrak{m}}$ is free as an $A_{\mathfrak{m}}$ -module for any maximal ideal $\mathfrak{m} \subset A$

Corollary. You have proved that **flat = locally free**.

Problem 3.[2pt] Show that

(1) P is projective $\Leftrightarrow \mathrm{Ext}_A^1(P, -) = 0 \Leftrightarrow \mathrm{Ext}_A^n(P, -) = 0$ for any $n \geq 1$
(2) I is injective $\Leftrightarrow \mathrm{Ext}_A^1(-, I) = 0 \Leftrightarrow \mathrm{Ext}_A^n(-, I) = 0$ for any $n \geq 1$

Problem 4.[20pt] Calculate:

(1) $\mathrm{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$
(2) $\mathrm{Ext}_{\mathbb{Z}/p^2\mathbb{Z}}^n(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$

The construction of a **direct limit**, described in the following exercise, often allows one to reduce a problem about modules to the finitely generated case.

Problem 5.[7pt] Read the construction of a direct limit in Exercise 2.14 (p.32) and the Universal property in Exercise 2.16 in [AM]. Then prove that any A-module is a direct limit of its finitely generated submodules (Exercise 2.17 in [AM]).

Problem 5 continued [This is an optional part, good for the review].

(1) Prove the Universal property of the direct limit (Ex. 2.16 in [AM])
(2) Prove that tensor product commutes with direct limits (Ex. 2.20 in [AM]).

If you could work through the following problem, it would indicate that you have digested the material very well and acquired some useful techniques. I'll be happy to discuss it.

Problem.[Optional] Let R be a commutative ring with identity, M, N be R-modules, and n be a non-negative integer. Show that the following are equivalent:

(1) $\mathrm{Tor}_n^R(M, N) = 0$
(2) $\mathrm{Tor}_n^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for any prime ideal $\mathfrak{p} \subset R$
(3) $\mathrm{Tor}_n^{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$ for any maximal ideal $\mathfrak{m} \subset R$

Problem.[Optional] Classify (up to isomorphism) finite dimensional representations of $\mathbb{Z}/p$ (over a field k).
Note: the answer should depend on the characteristic of the field k.