Homework 7 for 506, Spring 2009
due Friday, May 29

Throughout this homework, A is a commutative ring with identity.

The first three problems constitute a block with an ultimate goal to illustrate the non-uniqueness of the primary decomposition.

Problem 1. Let S be a multiplicatively closed subset of A. For an ideal $\mathfrak{a}$ denote by $S(\mathfrak{a})$ an ideal of A with is the restriction of the ideal $S^{-1}\mathfrak{a}$ of $S^{-1}A$ (we can think of it as $S(\mathfrak{a}) = A \cap S^{-1}\mathfrak{a}$). We always have an inclusion $\mathfrak{a} \subset S(\mathfrak{a})$, but if $\mathfrak{a}$ is not a prime ideal, then it can happen that $\mathfrak{a} \neq S(\mathfrak{a})$.

Now let $S = A - \mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset A$. The n^{ths} *symbolic power* of $\mathfrak{p}$ is the ideal

$$\mathfrak{p}^{(n)} \stackrel{\text{def}}{=} S(\mathfrak{p}^n).$$

(1) Show that $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary.
(2) Give an example of $\mathfrak{p}$ such that $\mathfrak{p}^n$ is a proper subset of $\mathfrak{p}^{(n)}$.

Problem 2. Let A be a Noetherian local ring, $\mathfrak{m}$ be the maximal ideal of A. Show that A is Artinian if and only if there exists r_0 such that $\mathfrak{m}^{r_0} = 0$.

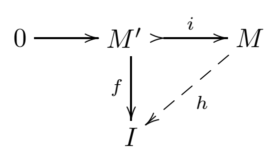
Problem 3. Let $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ be a minimal primary decomposition of the zero ideal in a Noetherian ring A, and let $\mathfrak{p}_i = \text{rad}(\mathfrak{q}_\text{i})$.

(1) Show that for any $i = 1, \ldots, n$ there exists $r_i > 0$ such that $\mathfrak{p}_i^{(r_i)} \subset q_i$.
(2) Let $\mathfrak{q}_i$ be an isolated component of the primary decomposition. Show that there exists r_i such that $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$ for all $r \geq r_i$.
(3) Let $\mathfrak{q}_i$ be an embedded component. Show that there are infinitely many r such that $\mathfrak{p}_i^{(r)}$ are all distinct.
(4) Conclude that if decomposition of zero above has an embedded $\mathfrak{p}_i$-primary component, then there are infinitely many distinct minimal primary decompositions which differ only in the $\mathfrak{p}_i$-primary component.

Problem 4. Show that P is a projective A-module if and only if $\text{Hom}_A(P, -)$ is an exact functor.

Problem 5. Let I be an A-module. Prove that the following are equivalent:

(1) For any injective homomorphism $i : M' \to M$ and any homomorphism $g : M' \to I$ there exists $h : M \to I$ such that the following diagram commutes:



$*$

(2) The functor $\text{Hom}_A(-, I) : \underline{\underline{A - \text{mod}}} \to \underline{\underline{A - \text{mod}}}$ is exact
(3) Any exact sequence $0 \to I \to M \to M'' \to 0$ splits

Hint: to prove 3) implies 1), take any diagram as in 1) and consider the module

$$I \oplus_{M'} M \stackrel{def}{=} \frac{I \oplus M}{\{(g(m'), -i(m')) \,|\, m' \in M'\}},$$

called the push-out of the diagram *:

$$\begin{array}{ccc} M' & \overset{i}{\rightarrowtail} & M \\ {\scriptstyle f}\downarrow & & \downarrow \\ I & \longrightarrow & I \oplus_{M'} M \end{array}$$

Show that the bottom horizontal map (induced by i) is injective; then apply 3) to that map.

Definition. A module satisfying one of these conditions is called **injective**.

In the next problem we shall describe injective modules over $\mathbb{Z}$. Note that this is a bit more involved than describing projective modules which are just $\mathbb{Z}^n$.

Problem 6.
I. Prove the **Baer's criterion** for injective modules: An A-module I is injective if and only if for any ideal $\mathfrak{a} \subset A$ and any map $f : \mathfrak{a} \to I$, the map f can be extended to $h : A \to I$:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathfrak{a} & \rightarrowtail & A \\ & & {\scriptstyle f}\downarrow & \swarrow_{h} & \\ & & I & & \end{array}$$

II. Show that an abelian group is injective (as a $\mathbb{Z}$-module) if an only if it is divisible. (An abelian group A is divisible if for any $a \in A$, and any $n \in \mathbb{Z}$ there exists $b \in A$ such that $a = nb$. For example, $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ are divisible.)