Homework 6 for 506, Spring 2009
due Friday, May 22

Problem 1.[20pt] Let A be a Noetherian ring, and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence of finitely generated A-modules. The sequence is called *split* if there exists a map $h : M'' \to M$:

$$0 \longrightarrow M' \xrightarrow{f} M \underset{h}{\overset{g}{\rightleftarrows}} M'' \longrightarrow 0\ ,$$

such that $g \circ h : M'' \to M''$ is the identity map. (In this case the module M splits as a direct sum: $M \simeq M' \oplus M''$). Show the sequence is split if and only if for any maximal ideal $\mathfrak{m} \subset A$ the short exact sequence

$$0 \longrightarrow M'_{\mathfrak{m}} \xrightarrow{f} M_{\mathfrak{m}} \xrightarrow{g} M''_{\mathfrak{m}} \longrightarrow 0$$

of $A_{\mathfrak{m}}$-modules is split.

For the next problem, you may use any of the results about Artinian rings that were stated without proof in class.

Problem 2.[20pt] Let k be a field, and A be a finitely generated k-algebra. Show that the following are equivalent:

(1) A is an Artinian ring;
(2) A is a finite k-algebra (that is, finite-dimensional as a vector space over k).

We know an algebra homomorphism $\phi : A \to B$ induces an injective map $\operatorname{Spec} B \to \operatorname{Spec} A$ if ϕ is onto. The question of when $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective is more subtle.

Problem 3.[10pt] Let $\phi : A \to B$ be a flat homomorphism (that is, ϕ makes B into a flat A-algebra). Prove that the following conditions are equivalent:

(1) For any A-module M, the map $M \to B \otimes_A M$ sending m to $1 \otimes m$ is injective;
(2) For an A-module M, $B \otimes_A M = 0$ implies $M = 0$;
(3) If $f : M \to N$ is an A-module map, and $1 \otimes f : B \otimes_A M \to B \otimes_A N$ is injective then f is injective.

Definition. A ring B satisfying the equivalent conditions from the previous problem is called a *faithfully flat* A-algebra.

Remark. Note that if B is faithfully flat, then the map ϕ is injective by the first condition; hence, we can identify A with a subring of B.

Problem 4.[20pt] Let $\phi : A \to B$ be flat. Show that the following conditions are equivalent:

(1) B is faithfully flat;
(2) $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective;
(3) For any maximal ideal $\mathfrak{m} \subset A$, $\mathfrak{m}B \neq B$ (the extension of $\mathfrak{m}$ to B is a proper ideal).